

Custodial symmetry violation in the Georgi-Machacek model

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Based on [B. Keeshan, HEL & T. Pilkington, arXiv:1807.11511](#)



Outline

Introduction

Georgi-Machacek model

Custodial symmetry violation

Our implementation

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Introduction

Can we constrain the possibility that “exotic” Higgs fields (isospin $> 1/2$) contribute to electroweak symmetry breaking?

Generically this is very strongly constrained by the ρ parameter:

$$\rho \equiv \frac{\text{weak neutral current}}{\text{weak charged current}} = \frac{(g^2 + g'^2)/M_Z^2}{g^2/M_W^2} = \frac{v_\phi^2 + a\langle X^0 \rangle^2}{v_\phi^2 + b\langle X^0 \rangle^2}$$

$$a = 4 [T(T + 1) - Y^2] c$$

$$b = 8Y^2$$

$$Q = T^3 + Y; \text{ SM doublet: } Y = 1/2$$

Expt: $\rho = 1.00037 \pm 0.00023$ (2016 PDG)

Need to do some model-building; otherwise $v_{\text{exotic}} \ll v_{\text{doublet}}$.

There are only two known approaches:

1) Use the septet $(T, Y) = (3, 2)$: $\rho = 1$ by accident!

Doublet $(\frac{1}{2}, \frac{1}{2}) +$ septet $(3, 2)$: **Scalar septet model**

Hisano & Tsumura, 1301.6455; Kanemura, Kikuchi & Yagyu, 1301.7303

2) Use global $SU(2)_L \times SU(2)_R$ imposed on the scalar potential

Global $SU(2)_L \times SU(2)_R \rightarrow$ custodial $SU(2)$ ensures tree-level $\rho = 1$

Doublet + triplets $(1, 0) + (1, 1)$: **Georgi-Machacek model**

Georgi & Machacek 1985; Chanowitz & Golden 1985

Doublet + quartets $(\frac{3}{2}, \frac{1}{2}) + (\frac{3}{2}, \frac{3}{2})$: **Generalized Georgi-**

Doublet + quintets $(2, 0) + (2, 1) + (2, 2)$: **Machacek models**

Doublet + sextets $(\frac{5}{2}, \frac{1}{2}) + (\frac{5}{2}, \frac{3}{2}) + (\frac{5}{2}, \frac{5}{2})$:

Galison 1984; Robinett 1985; HEL 1999; Chang et al 2012; HEL & Rentala 2015

Larger than sextets \rightarrow too many large multiplets, violates perturbativity

Can also have duplications, combinations \rightarrow ignore that here.

Both approaches have theoretical “issues”:

1) Can't give the septet a vev through spontaneous breaking without generating a physical massless Goldstone boson.

Have to couple it to the SM doublet through a dimension-7 $X\Phi^*\Phi^5$ term [Hisano & Tsumura 2013](#)

Need the UV completion to be nearby!

2) Global $SU(2)_L \times SU(2)_R$ is broken by gauging hypercharge.

[Gunion, Vega & Wudka 1991](#)

Special relations among params of *full* gauge-invariant scalar potential can only hold at one energy scale: violated by running due to hypercharge. [Garcia-Pepin, Gori, Quiros, Vega, Vega-Morales, Yu 2014](#)

Need the UV completion to be nearby!

This talk: quantify (2) in the Georgi-Machacek model.

Georgi-Machacek model [Georgi & Machacek 1985](#); [Chanowitz & Golden 1985](#)

SM Higgs (bi-)doublet + triplets (1, 0) + (1, 1) in a **bi-triplet**:

$$\Phi = \begin{pmatrix} \phi^{0*} & \phi^+ \\ -\phi^{+*} & \phi^0 \end{pmatrix} \quad X = \begin{pmatrix} \chi^{0*} & \xi^+ & \chi^{++} \\ -\chi^{+*} & \xi^0 & \chi^+ \\ \chi^{++*} & -\xi^{+*} & \chi^0 \end{pmatrix}$$

Global $SU(2)_L \times SU(2)_R \rightarrow$ custodial symmetry $\langle \chi^0 \rangle = \langle \xi^0 \rangle \equiv v_\chi$

Most general scalar potential invariant under $SU(2)_L \times SU(2)_R$:

$$\begin{aligned} V(\Phi, X) = & \frac{\mu_2^2}{2} \text{Tr}(\Phi^\dagger \Phi) + \frac{\mu_3^2}{2} \text{Tr}(X^\dagger X) + \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 \\ & + \lambda_2 \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(X^\dagger X) + \lambda_3 \text{Tr}(X^\dagger X X^\dagger X) \\ & + \lambda_4 [\text{Tr}(X^\dagger X)]^2 - \lambda_5 \text{Tr}(\Phi^\dagger \tau^a \Phi \tau^b) \text{Tr}(X^\dagger t^a X t^b) \\ & - M_1 \text{Tr}(\Phi^\dagger \tau^a \Phi \tau^b) (UXU^\dagger)_{ab} - M_2 \text{Tr}(X^\dagger t^a X t^b) (UXU^\dagger)_{ab} \end{aligned}$$

9 parameters, 2 fixed by G_F and $m_h \rightarrow 7$ free parameters. [Aoki & Kanemura, 0712.4053](#)

[Chiang & Yagyu, 1211.2658](#); [Chiang, Kuo & Yagyu, 1307.7526](#)

[Hartling, Kumar & HEL, 1404.2640](#)

Georgi-Machacek model

Georgi & Machacek 1985; Chanowitz & Golden 1985

SM Higgs (bi-)doublet + triplets (1,0) + (1,1) in a **bi-triplet**:

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Global $SU(2)_L \times SU(2)_R \rightarrow$ custodial symmetry $\langle \chi^0 \rangle = \langle \xi^0 \rangle \equiv v_\chi$

Physical spectrum:

Bi-doublet: $2 \otimes 2 \rightarrow 1 \oplus 3$

Bi-triplet: $3 \otimes 3 \rightarrow 1 \oplus 3 \oplus 5$

- Two custodial singlets mix $\rightarrow h^0, H^0$ m_h, m_H , angle α
Usually identify $h^0 = h(125)$

- Two custodial triplets mix $\rightarrow (H_3^+, H_3^0, H_3^-)$ m_3 + Goldstones
Phenomenology very similar to H^\pm, A^0 in 2HDM Type I, $\tan \beta \rightarrow \cot \theta_H$

- Custodial fiveplet $(H_5^{++}, H_5^+, H_5^0, H_5^-, H_5^{--})$ m_5
Fermiophobic; $H_5 VV$ couplings $\propto s_H \equiv \sqrt{8}v_\chi/v_{SM}$
 $s_H^2 \equiv$ exotic fraction of M_W^2, M_Z^2

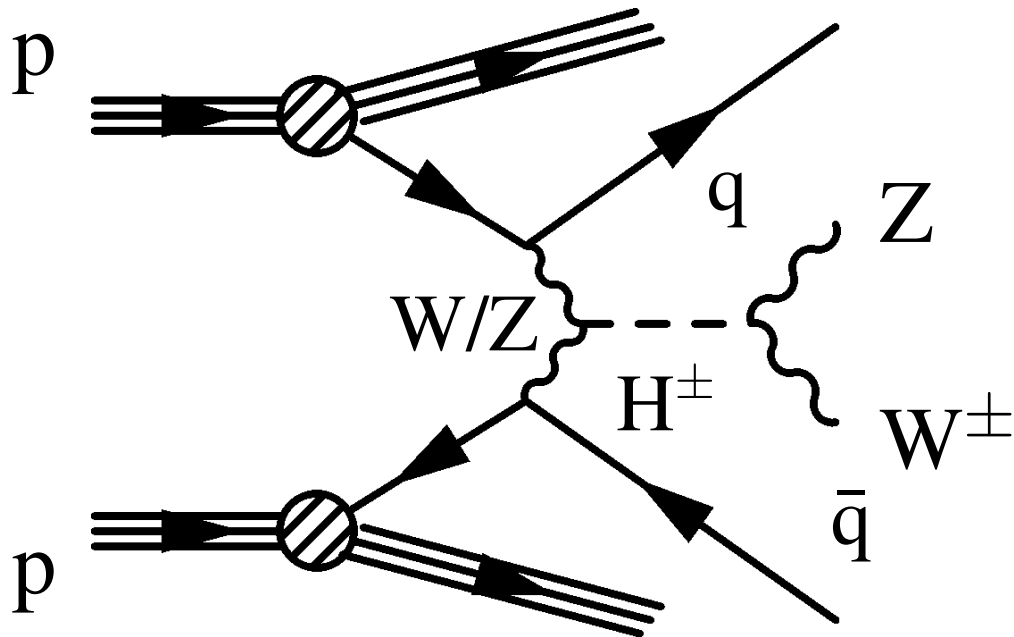
Smoking-gun processes:

$$\text{VBF} \rightarrow H_5^{\pm\pm} \rightarrow W^\pm W^\pm$$

VBF + like-sign dileptons + MET

$$\text{VBF} \rightarrow H_5^\pm \rightarrow W^\pm Z$$

VBF + $q\bar{q}l\bar{l}$; VBF + $3l$ + MET



Cross section $\propto s_H^2 \equiv$ fraction of M_W^2, M_Z^2 due to exotic scalars

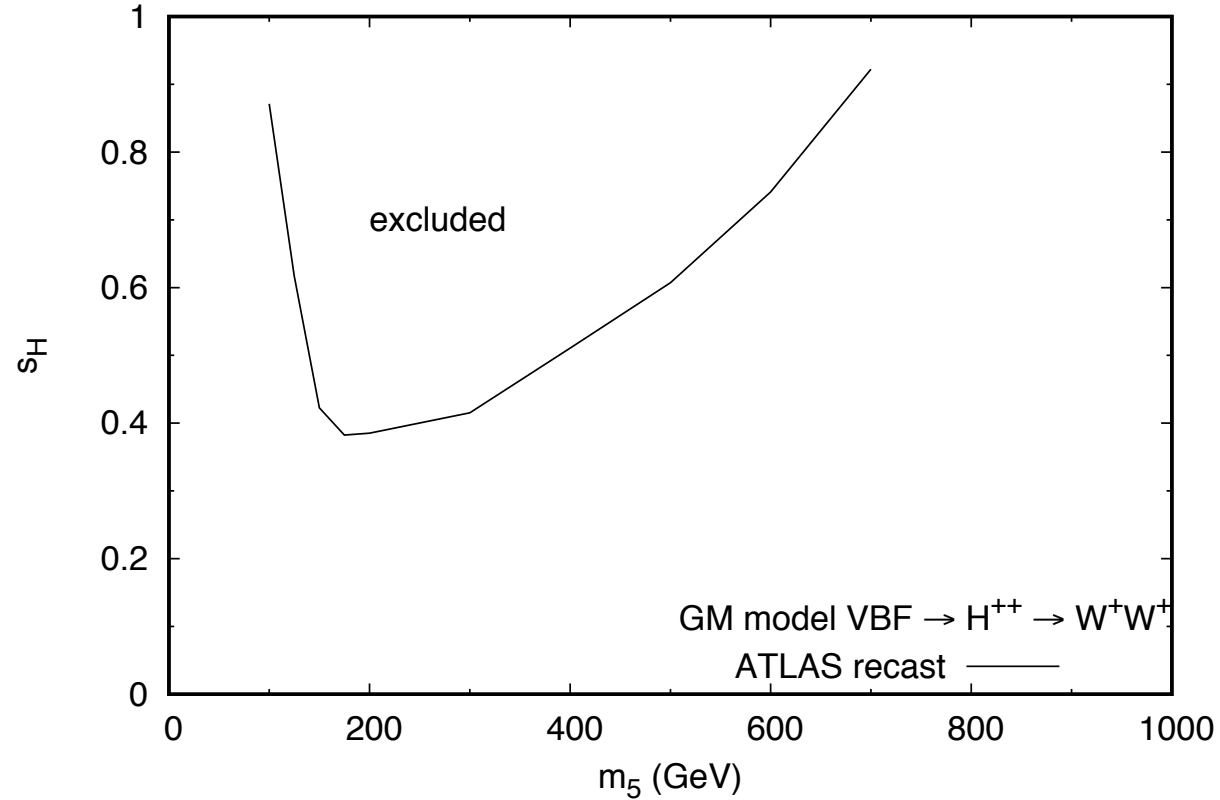
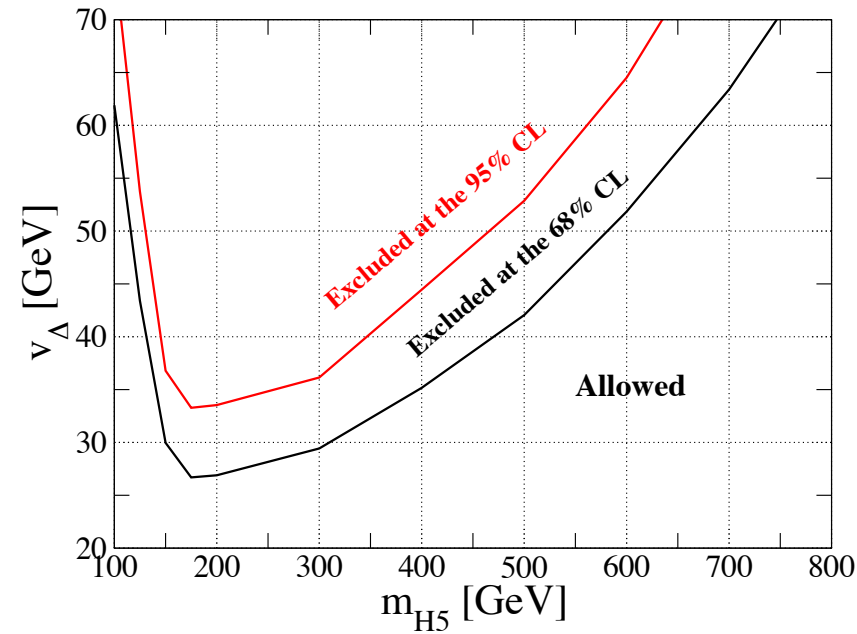
Searches

SM $VBF \rightarrow W^\pm W^\pm \rightarrow \ell^\pm \ell^\pm + MET$ cross section measurement

ATLAS Run 1 1405.6241, PRL 2014

Recast to constrain $VBF \rightarrow H_5^{\pm\pm} \rightarrow W^\pm W^\pm \rightarrow \ell^\pm \ell^\pm + MET$

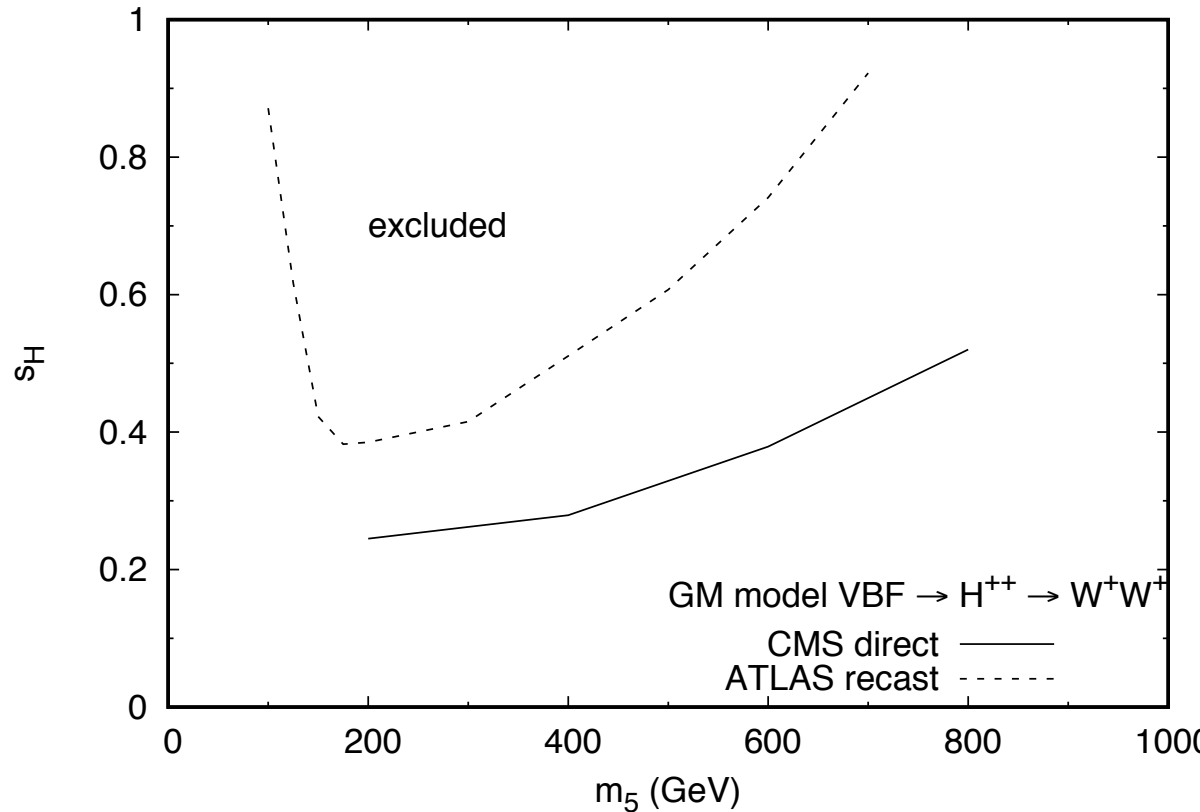
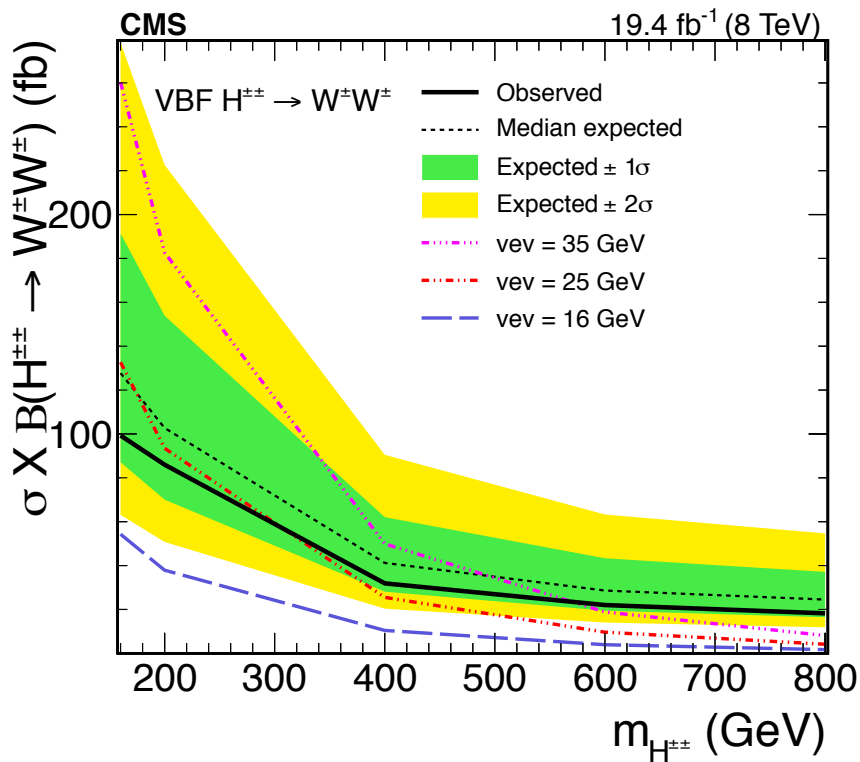
Chiang, Kanemura, Yagyu, 1407.5053



Searches

VBF $H_5^{\pm\pm} \rightarrow W^\pm W^\pm \rightarrow \ell^\pm \ell^\pm + \text{MET}$ (CMS Run 1)

CMS 1410.6315, PRL 2015



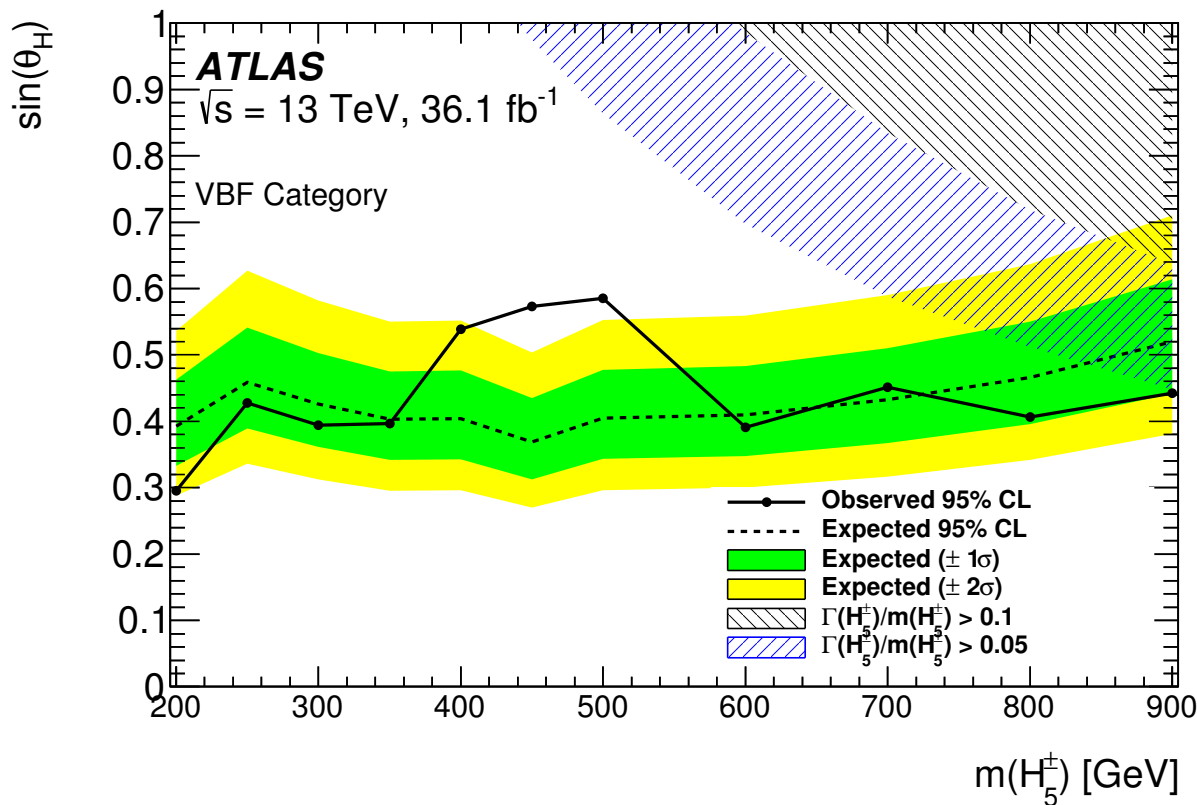
Translated using VBF $\rightarrow H^{\pm\pm}$ cross sections from [LHCHXSWG-2015-001](#) (M. Zaro + HEL)

Searches

New this summer!

VBF $H_5^\pm \rightarrow W^\pm Z \rightarrow \ell^\pm \ell^+ \ell^- + \text{MET}$ (ATLAS Run 2)

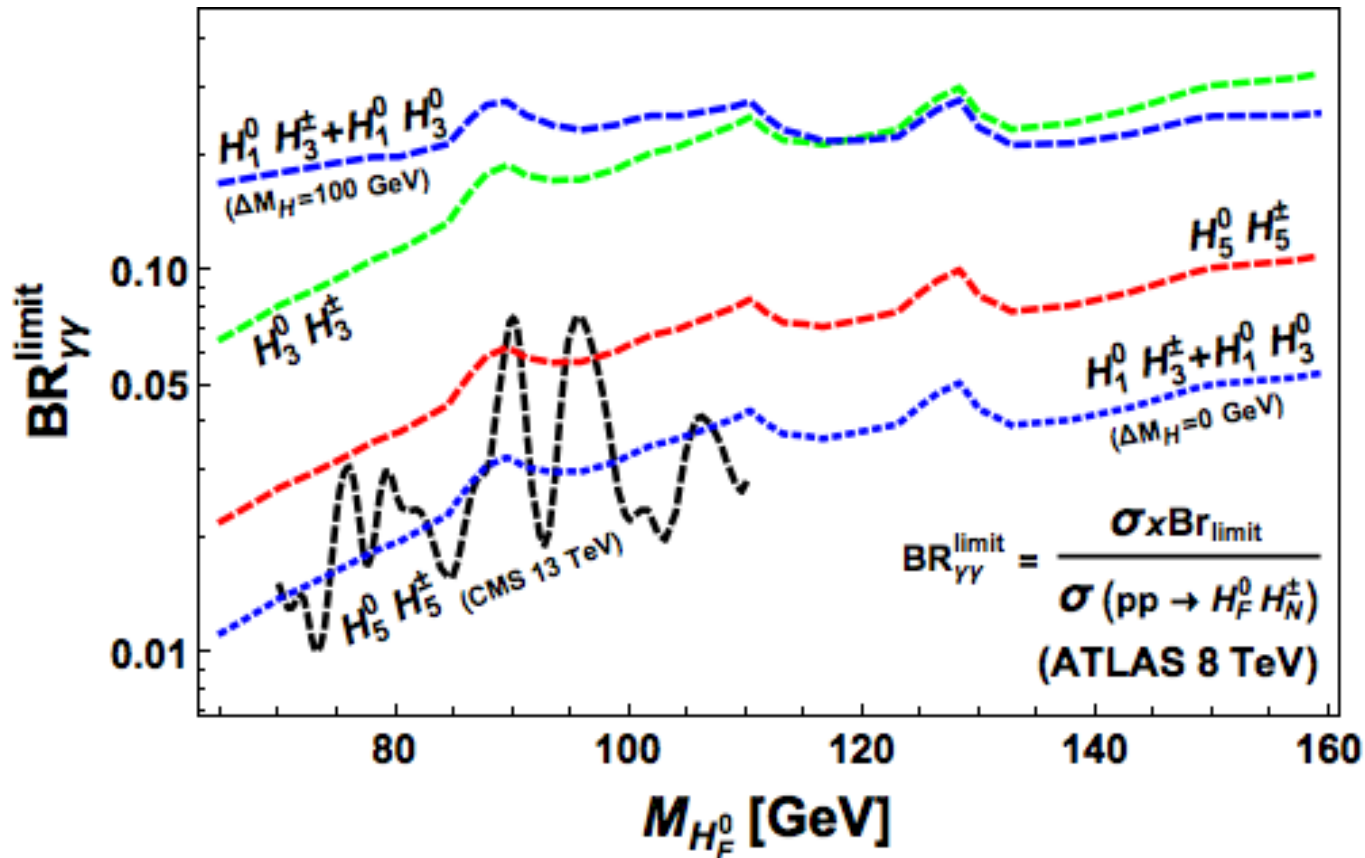
ATLAS 1806.01532



Stronger upper bound on s_H for $m_5 \in (700, 900)$ GeV compared to $H_5^{\pm\pm}$

Searches: $H_5^0 \rightarrow \gamma\gamma$ at low mass

Drell-Yan $pp \rightarrow H_5^0 H_5^\pm$ depends only on m_5 and gauge couplings!



Vega, Vega-Morales & Xie, 1805.01970

If W loop contribution dominates $H_5^0 \rightarrow \gamma\gamma, Z\gamma$, tree and loop decays scale the same way with s_H and $m_5 \lesssim 110$ GeV is excluded.

Custodial symmetry violation in the GM model: a long history

Gunion, Vega & Wudka 1991 showed that computing the T parameter in the GM model yields infinity due to an **uncancelled UV divergence** caused by hypercharge violating the custodial symmetry at 1-loop. Full gauge-invariant but $SU(2)_L \times SU(2)_R$ -violating scalar potential yields the needed counterterm.

Englert, Re & Spannowsky 1302.6505 applied S, T parameter constraints by subtracting a counterterm for T (just the divergent part? not clear)

Chiang, Kuo & Yagyu 1804.02633 calculated 1-loop renormalized predictions for h couplings in GM model and used measured T parameter as input to fix the relevant custodial-symmetry-violating counterterm

Blasi, De Curtis & Yagyu 1704.08512 computed the RGEs and studied custodial violation from running up from custodial-symmetric theory at the weak scale (RGEs independently checked by us)

Full gauge-invariant potential:

$$\begin{aligned}
V(\phi, \chi, \xi) = & \tilde{\mu}_2^2 \phi^\dagger \phi + \tilde{\mu}_3'^2 \chi^\dagger \chi + \frac{\tilde{\mu}_3^2}{2} \xi^\dagger \xi \\
& + \tilde{\lambda}_1 (\phi^\dagger \phi)^2 + \tilde{\lambda}_2 |\tilde{\chi}^\dagger \chi|^2 + \tilde{\lambda}_3 (\phi^\dagger \tau^a \phi) (\chi^\dagger t^a \chi) \\
& + [\tilde{\lambda}_4 (\tilde{\phi}^\dagger \tau^a \phi) (\chi^\dagger t^a \xi) + \text{h.c.}] + \tilde{\lambda}_5 (\phi^\dagger \phi) (\chi^\dagger \chi) \\
& + \tilde{\lambda}_6 (\phi^\dagger \phi) (\xi^\dagger \xi) + \tilde{\lambda}_7 (\chi^\dagger \chi)^2 + \tilde{\lambda}_8 (\xi^\dagger \xi)^2 \\
& + \tilde{\lambda}_9 |\chi^\dagger \xi|^2 + \tilde{\lambda}_{10} (\chi^\dagger \chi) (\xi^\dagger \xi) \\
& - \frac{1}{2} [\tilde{M}_1' \phi^\dagger \Delta_2 \tilde{\phi} + \text{h.c.}] + \frac{\tilde{M}_1}{\sqrt{2}} \phi^\dagger \Delta_0 \phi - 6 \tilde{M}_2 \chi^\dagger \bar{\Delta}_0 \chi
\end{aligned}$$

where

$$\begin{aligned}
\Delta_2 & \equiv \sqrt{2} \tau^a U_{ai} \chi_i = \begin{pmatrix} \chi^+ / \sqrt{2} & -\chi^{++} \\ \chi^0 & -\chi^+ / \sqrt{2} \end{pmatrix}, \\
\Delta_0 & \equiv \sqrt{2} \tau^a U_{ai} \xi_i = \begin{pmatrix} \xi^0 / \sqrt{2} & -\xi^+ \\ -\xi^{+*} & -\xi^0 / \sqrt{2} \end{pmatrix}, \\
\bar{\Delta}_0 & \equiv -t^a U_{ai} \xi_i = \begin{pmatrix} -\xi^0 & \xi^+ & 0 \\ \xi^{+*} & 0 & \xi^+ \\ 0 & \xi^{+*} & \xi^0 \end{pmatrix}.
\end{aligned}$$

Minimize potential, compute mass matrices, etc.

16 Lagrangian parameters compared to 9 in original GM model:
 Matching gauge-invariant potential to original GM model yields

$$\begin{aligned}
 \tilde{\mu}_2^2 &= \mu_2^2 & \tilde{\lambda}_6 &= 2\lambda_2 \\
 \tilde{\mu}_3'^2 &= \mu_3^2 & \tilde{\lambda}_7 &= 2\lambda_3 + 4\lambda_4 \\
 \tilde{\mu}_3^2 &= \mu_3^2 & \tilde{\lambda}_8 &= \lambda_3 + \lambda_4 \\
 \tilde{\lambda}_1 &= 4\lambda_1 & \tilde{\lambda}_9 &= 4\lambda_3 \\
 \tilde{\lambda}_2 &= 2\lambda_3 & \tilde{\lambda}_{10} &= 4\lambda_4 \\
 \tilde{\lambda}_3 &= -2\lambda_5 & \tilde{M}_1' &= M_1 \\
 \tilde{\lambda}_4 &= -\sqrt{2}\lambda_5 & \tilde{M}_1 &= M_1 \\
 \tilde{\lambda}_5 &= 4\lambda_2 & \tilde{M}_2 &= M_2
 \end{aligned}$$

RGEs with $g' = 0$ preserve these relations.

Keeping $g' \neq 0$ violates these relations and introduces custodial symmetry violation through the RGE running.

Our implementation

Basic idea:

- Assume custodial symmetry at some high scale Λ
(accidental $SU(2)_L \times SU(2)_R$ coming from UV completion e.g. composite Higgs)
- Run down to weak scale \Rightarrow custodial violation generated
(1-loop RGEs, tree-level matching \equiv leading log approximation)
Measured value of ρ will put an **upper bound** on scale Λ
- Subject to ρ constraint (and perturbativity at Λ), quantify maximum allowed custodial symmetry violation and its phenomenological consequences

Our implementation

Details:

- Start with a benchmark scenario at the weak scale* (for concreteness, and to get G_F , m_h close to their correct values)

* “weak scale” = m_5

H5plane benchmark (introduced by HXSWG for H_5 LHC searches)

Fixed Parameters	Variable Parameters	Dependent Parameters
$G_F = 1.1663787 \times 10^{-5} \text{ GeV}^{-2}$	$m_5 \in [200, 3000] \text{ GeV}$	$\lambda_2 = 0.4m_5/(1000 \text{ GeV})$
$m_h = 125 \text{ GeV}$	$s_H \in (0, 1)$	$M_1 = \sqrt{2}s_H(m_5^2 + v^2)/v$
$\lambda_3 = -0.1$		$M_2 = M_1/6$
$\lambda_4 = 0.2$		

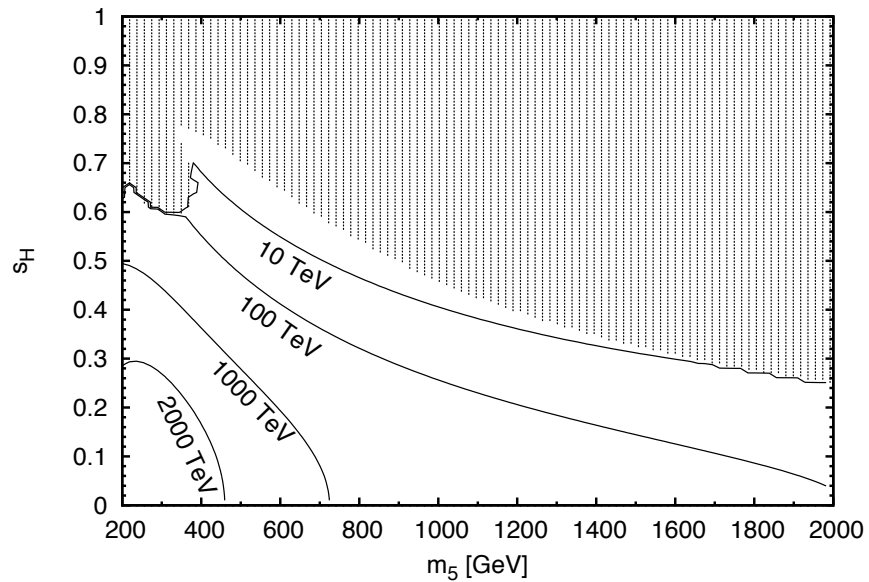
- Run up with $g' = 0$ (custodial symmetric!) to some scale Λ ;
check perturbativity of quartic couplings (avoid Landau pole)
 \Rightarrow upper bound on Λ to avoid perturbativity violation

Our implementation

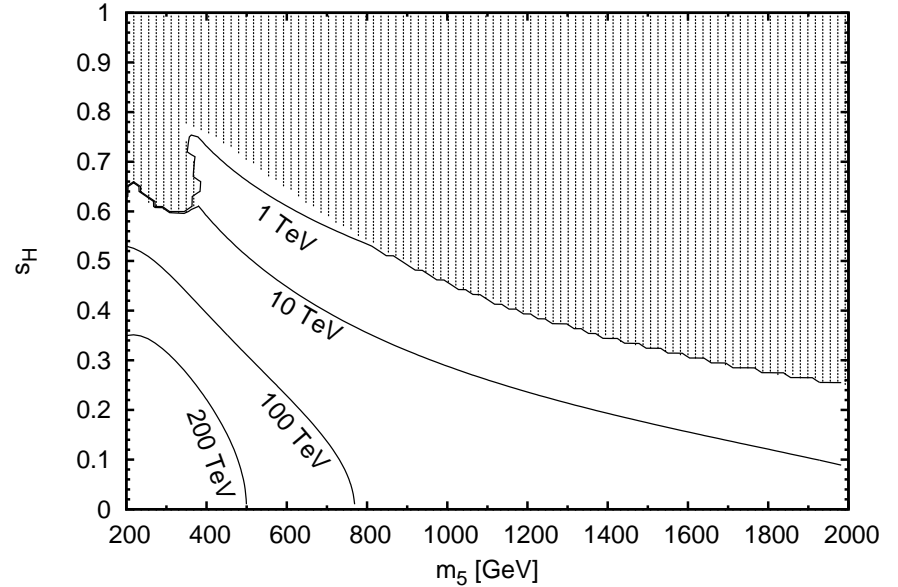
Details (continued):

- Run back down with $g' \neq 0$ to get the custodial-violating Lagrangian parameters at the weak scale
- Compute vevs $\rightarrow G_F$ and mass matrices $\rightarrow m_h$; adjust original weak-scale inputs and iterate until these match experiment in custodial violating theory
- Compute ρ ; adjust upper bound on Λ if necessary
 $\rho = 1.00037 \pm 0.00023$ (2016 PDG) [require within $\pm 2\sigma$]
- Compute weak-scale predictions for custodial-violating observables (λ_{WZ}^h , mass splittings, mixings)

Results (within H5plane benchmark): cutoff scale



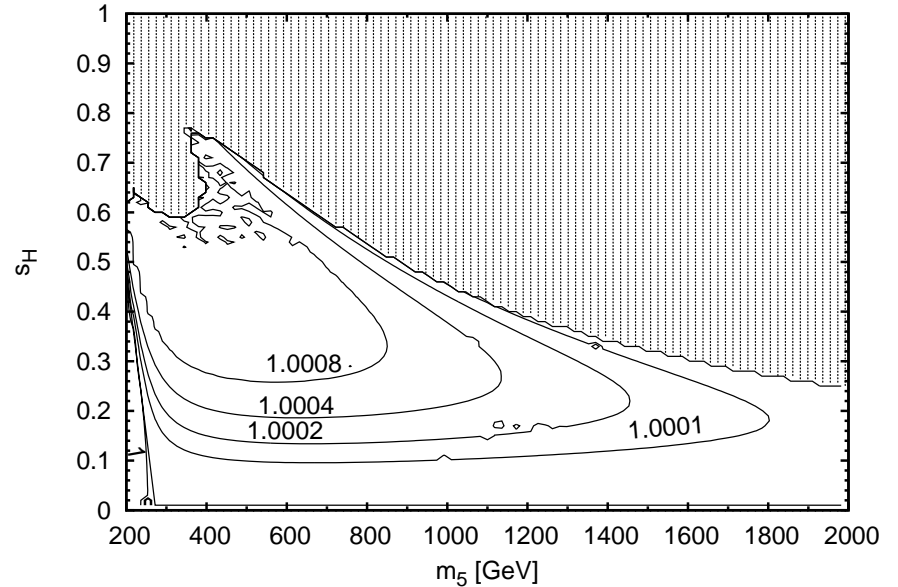
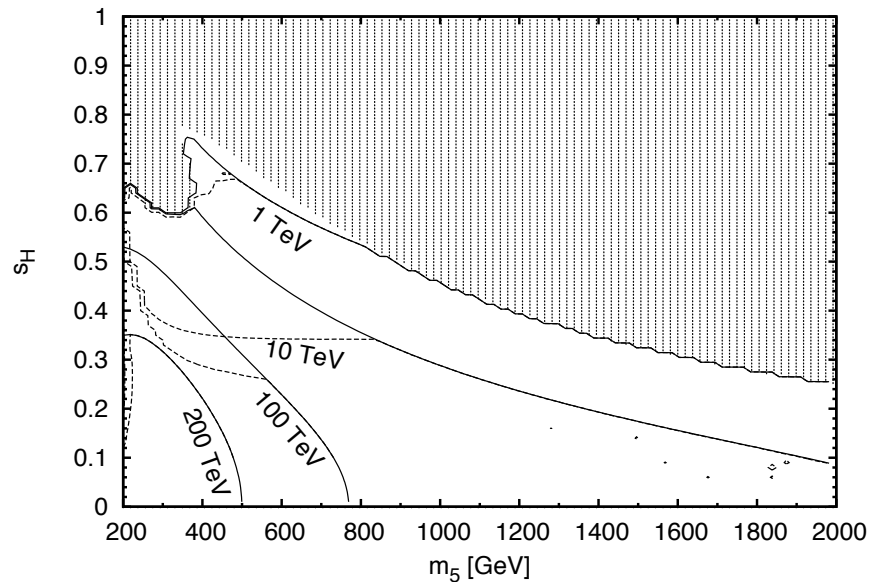
Left: Scale of Landau pole



Right: Highest scale at which perturbative unitarity constraints on custodial-symmetric λ_i remain satisfied

UV completion must appear below 10s to 100s of TeV

Results (within H5plane benchmark): ρ parameter



Left: Maximum cutoff scale including ρ parameter constraint (dashed) + perturbative unitarity (solid)

Right: Weak-scale value of ρ , for Λ as large as possible

ρ samples full 2σ allowed range

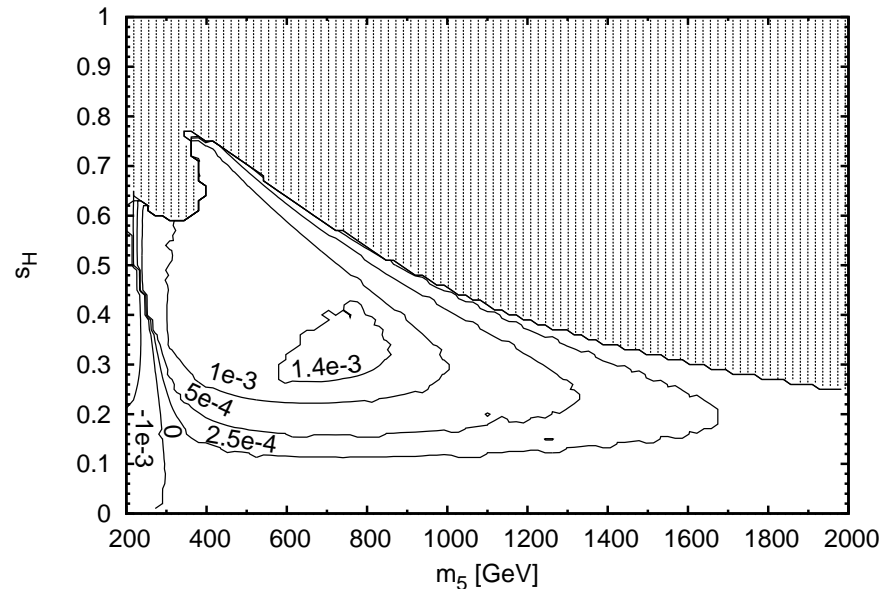
$\Delta\rho$ is positive in most of H5plane benchmark parameter space

Results (within H5plane benchmark): hWW/hZZ

Test of custodial symmetry violation in h_{125} couplings:

$$\lambda_{WZ}^h \equiv \frac{\kappa_W^h}{\kappa_Z^h}$$

Plot: $\delta\lambda_{WZ}^h \equiv \lambda_{WZ}^h - 1$
for Λ as large as possible



Deviation from SM prediction ($\lambda_{WZ}^h = 1$) at most half a percent

Current LHC precision: $\lambda_{WZ}^h = 0.88_{-0.09}^{+0.10}$ ATLAS + CMS Run 1, 1606.02266

Future precision based on hWW and hZZ projections:

HL-LHC: a few percent

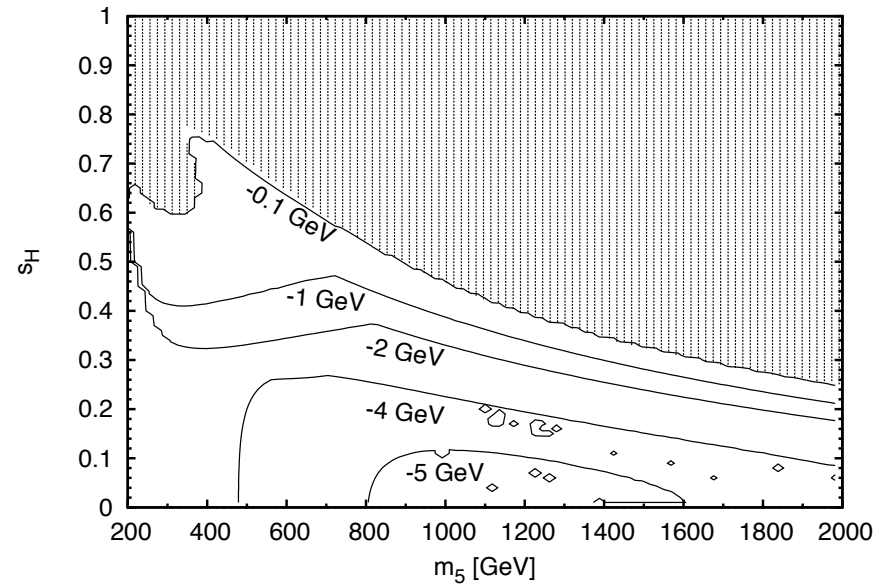
ILC: about half a percent

FCC-ee: about 0.2 percent

Results (within H5plane benchmark): mass splittings

Plot: $m_{H_3^\pm} - m_{H_3^0}$
for Λ as large as possible

(negative values: H_3^\pm is lighter)



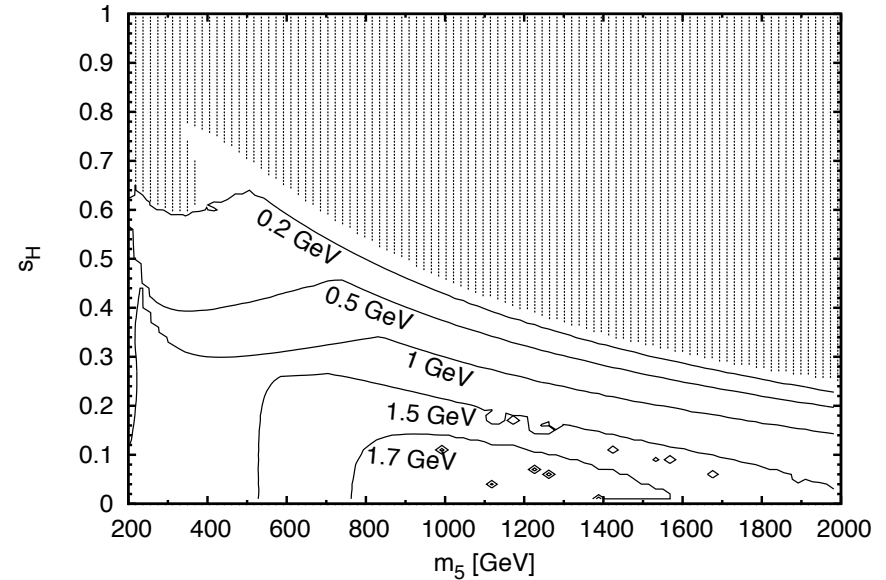
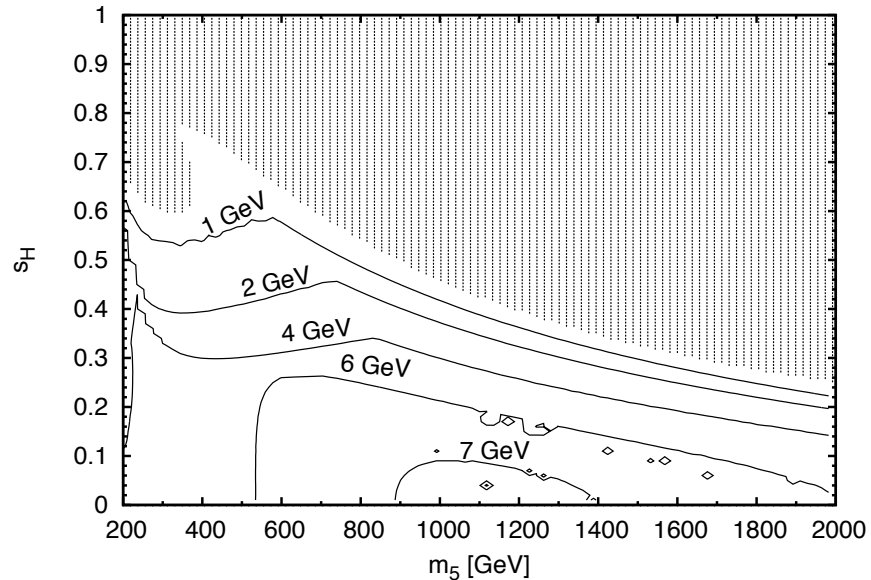
Custodial-violating mass splitting of $H_3^{0,\pm}$ is at most 5.3 GeV.
 $m_{H_3^0} > m_{H_3^\pm}$ everywhere in H5plane benchmark.

Measurement prospects: $H_3^0 \rightarrow b\bar{b}, t\bar{t}$; $H_3^\pm \rightarrow t\bar{b}$

Couplings as in Type-I 2HDM: down-type decays not enhanced

Mass splitting too small to detect at LHC

Results (within H5plane benchmark): mass splittings



Left: $m_{H_5^{\pm\pm}} - m_{H_5^0}$
for Λ as large as possible

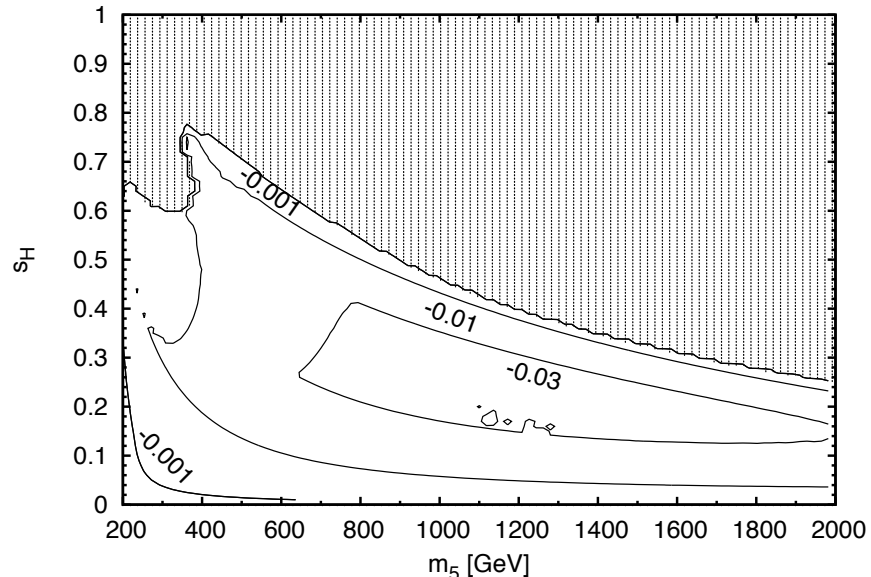
Right: $m_{H_5^{\pm}} - m_{H_5^0}$

Custodial-violating mass splitting of $H_5^{0,\pm,\pm\pm}$ is at most 7.2 GeV.
 $m_{H_5^{\pm\pm}} > m_{H_5^{\pm}} > m_{H_5^0}$ everywhere in H5plane benchmark.

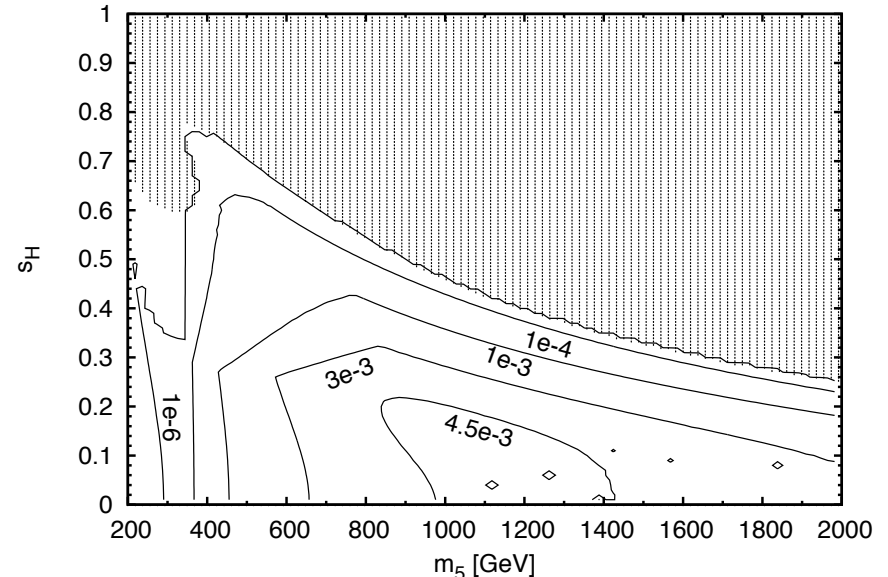
Decays are to VV – similar challenges to detect small mass splittings at LHC.

Results (within H5plane benchmark): mixing

Custodial sym violation mixes doublet into fermiophobic H_5



Left: induced κ_f for H_5^0
for Λ as large as possible



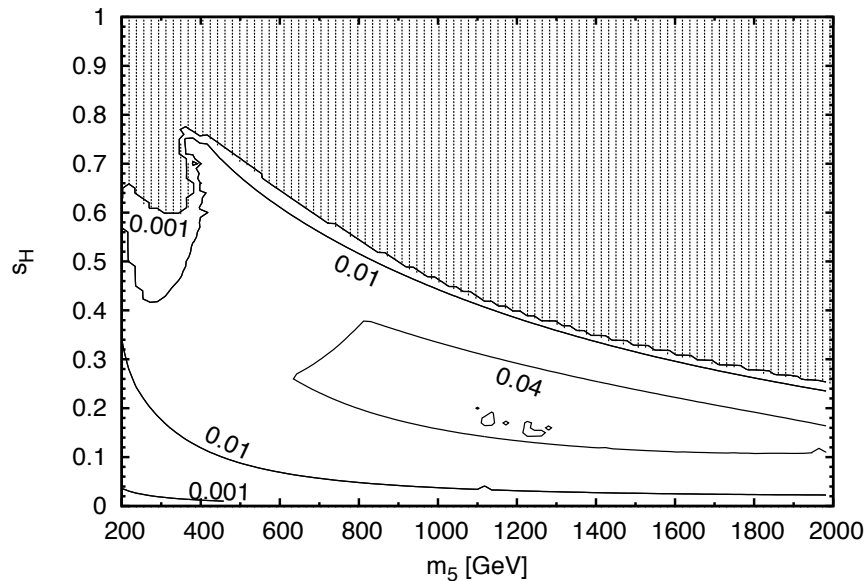
Right: $BR(H_5^0 \rightarrow f\bar{f})$

Custodial-violation-induced BR of H_5^0 to fermions ($t\bar{t}$) reaches almost half a percent in H5plane benchmark ($m_5 > 200$ GeV).

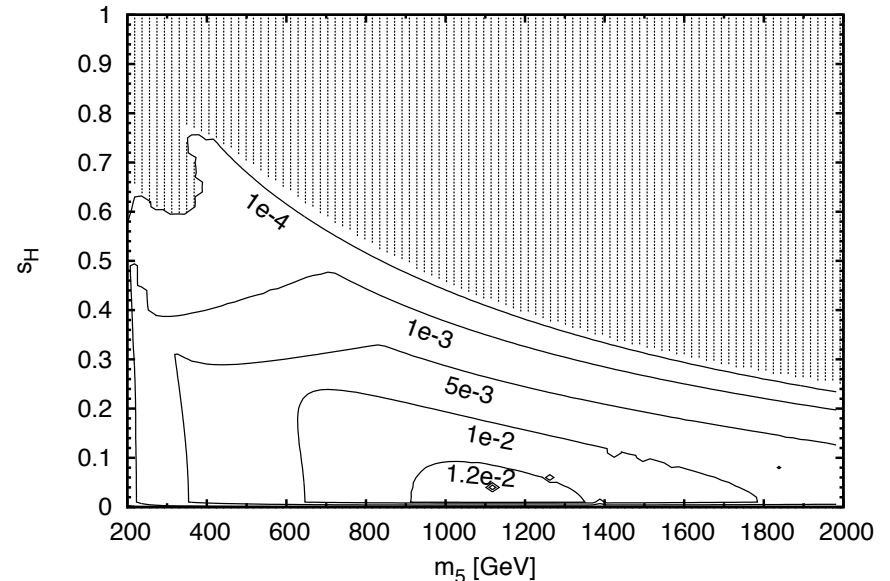
Effect at low mass $< 2M_W$ may be much more interesting:
competition with powerful $H_5^0 \rightarrow \gamma\gamma$ channel \Rightarrow future work

Results (within H5plane benchmark): mixing

Custodial sym violation mixes doublet into fermiophobic H_5



Left: induced κ_f for H_5^\pm
for Λ as large as possible



Right: $\text{BR}(H_5^+ \rightarrow f\bar{f})$

Custodial-violation-induced BR of H_5^+ to fermions ($t\bar{b}$) reaches 1.2% in H5plane benchmark ($m_5 > 200$ GeV).

At low mass $< M_W + M_Z$ this can compete with $W\gamma$ decay.

Conclusions and outlook

Hypercharge interactions violate custodial symmetry in the GM model beyond tree level.

We studied the impact of this assuming a custodial-symmetric theory at some high scale Λ and running down to the weak scale.

For this first pass we used the H5plane benchmark ($m_5 > 200$ GeV)

Main results:

- UV completion must lie below 10s to 100s of TeV
forced by perturbative unitarity + measured ρ parameter
- Custodial-violating effects are small! (too small to see at LHC)
assumption of custodial-symmetric GM is good for LHC searches

Custodial-violation-induced fermion couplings of otherwise fermiophobic H_5 may become important for masses below $2M_W$, where tree-level decays go offshell and loop decays become important.

Competition with powerful $H_5^0 \rightarrow \gamma\gamma$ channel? \Rightarrow future work

BACKUP SLIDES

Most general $SU(2)_L \times SU(2)_R$ -invariant scalar potential: Aoki & Kanemura, 0712.4053

Chiang & Yagyu, 1211.2658; Chiang, Kuo & Yagyu, 1307.7526

Hartling, Kumar & HEL, 1404.2640

$$\begin{aligned}
 V(\Phi, X) = & \frac{\mu_2^2}{2} \text{Tr}(\Phi^\dagger \Phi) + \frac{\mu_3^2}{2} \text{Tr}(X^\dagger X) + \lambda_1 [\text{Tr}(\Phi^\dagger \Phi)]^2 \\
 & + \lambda_2 \text{Tr}(\Phi^\dagger \Phi) \text{Tr}(X^\dagger X) + \lambda_3 \text{Tr}(X^\dagger X X^\dagger X) \\
 & + \lambda_4 [\text{Tr}(X^\dagger X)]^2 - \lambda_5 \text{Tr}(\Phi^\dagger \tau^a \Phi \tau^b) \text{Tr}(X^\dagger t^a X t^b) \\
 & - M_1 \text{Tr}(\Phi^\dagger \tau^a \Phi \tau^b) (UXU^\dagger)_{ab} - M_2 \text{Tr}(X^\dagger t^a X t^b) (UXU^\dagger)_{ab}
 \end{aligned}$$

9 parameters, 2 fixed by M_W and $m_h \rightarrow$ free parameters are $m_H, m_3, m_5, v_\chi, \alpha$ plus two triple-scalar couplings.

Dimension-3 terms usually omitted by imposing Z_2 sym. on X .

These dim-3 terms are essential for the model to possess a decoupling limit!

$(UXU^\dagger)_{ab}$ is just the matrix X in the Cartesian basis of $SU(2)$, found using

$$U = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}$$

Fields and vevs for full gauge-invariant potential:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \quad \chi = \begin{pmatrix} \chi^{++} \\ \chi^+ \\ \chi^0 \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi^+ \\ \xi^0 \\ -\xi^{+*} \end{pmatrix}, \quad (1)$$

$$\begin{aligned} \tilde{\phi} &\equiv C_2 \phi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^{+*} \end{pmatrix} \\ \tilde{\chi} &\equiv C_3 \chi^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \chi^* = \begin{pmatrix} \chi^{0*} \\ -\chi^{+*} \\ \chi^{++*} \end{pmatrix}. \end{aligned} \quad (2)$$

$$\phi^0 \rightarrow \frac{\tilde{v}_\phi}{\sqrt{2}} + \frac{\phi^{0,r} + i\phi^{0,i}}{\sqrt{2}}, \quad \chi^0 \rightarrow \tilde{v}_\chi + \frac{\chi^{0,r} + i\chi^{0,i}}{\sqrt{2}}, \quad \xi^0 \rightarrow \tilde{v}_\xi + \xi^{0,r}. \quad (3)$$

$$\rho = \frac{\tilde{v}_\phi^2 + 4\tilde{v}_\chi^2 + 4\tilde{v}_\xi^2}{\tilde{v}_\phi^2 + 8\tilde{v}_\chi^2} = \frac{v^2}{v^2 + 4(\tilde{v}_\chi^2 - \tilde{v}_\xi^2)}. \quad (4)$$