Carleton University Physics Department
PHYS 4708 (Winter 2015, H. Logan)
Midterm exam

This exam is closed-book and -notes. The three questions will be weighted equally.

1. Consider a system of two spin-1 particles with total angular momentum $\vec{J} = \vec{S}_1 + \vec{S}_2$. In certain materials, various effects conspire to create an effective interaction $V = -\lambda \vec{S}_1 \cdot \vec{S}_2$ between neighbouring spins, where λ is a positive number.

   (a) Compute the energy shifts caused by V. Use the appropriate basis in accordance with degenerate perturbation theory.

   (b) Consider the set of states with $j = 2$ and $m_j = -j, \ldots, j$. Which of these states are eigenstates of $S_{1z}S_{2z}$, and what are the corresponding eigenvalues? For the state(s) that are not eigenstates of $S_{1z}S_{2z}$, compute the expectation value of $S_{1z}S_{2z}$. (Use the table of Clebsch-Gordan coefficients provided.)

   (c) Use the results of parts (a) and (b) to determine the expectation value of $(S_{1x}S_{2x}+S_{1y}S_{2y})$ in each of the states with $j = 2$.

2. Consider a one-dimensional harmonic oscillator with unperturbed Hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2, \tag{1}$$

   with eigenstates $|n\rangle$ whose energies are $E_n^{(0)} = \hbar\omega(n + 1/2)$. It is subject to a perturbing Hamiltonian

$$H_1 = \lambda x^2. \tag{2}$$

   (a) Compute the first-order energy shift $E_n^{(1)}$ of level n of this harmonic oscillator. You can use the expression for x in terms of raising and lowering operators,

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \tag{3}$$

   where $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

   (b) Compute the first-order correction to the energy eigenstates (i.e., find the new eigenstates $|\psi_n\rangle$ in terms of the original eigenstates $|\phi_n\rangle \equiv |n\rangle$).

   (c) Compute the second-order energy shift $E_n^{(2)}$ of level n.

   (d) This problem can be solved exactly by writing

$$H = H_0 + H_1 = \frac{p^2}{2m} + \frac{1}{2} m\omega'^2 x^2. \tag{4}$$

   Find ω' in terms of ω and λ and write down the exact perturbed energies E_n'. Do a series expansion of ω' out to second order in λ and check that these terms agree with your results for $E_n^{(1)}$ and $E_n^{(2)}$ found above.

continued....

3. Consider a system of two spin-1 particles. The total angular momentum is $\vec{J} = \vec{S}_1 + \vec{S}_2$.

   (a) Make a table showing the combinations of m_{s1} and m_{s2} that can contribute to each m_j value. What are the allowed values of j?

   (b) Write down the state $|j = 2, m_j = 2\rangle$ in terms of the $|m_{s1}, m_{s2}\rangle$ basis states. Show that this state is an eigenstate of the exchange operator P_{12}, which swaps particle 1 for particle 2, and find its eigenvalue. Then use the fact that $[J_\pm, P_{12}] = 0$ (first convince yourself that this is true!) to write down the states $|j = 2, m_j = 1\rangle$ and $|j = 2, m_j = -1\rangle$ in terms of the $|m_{s1}, m_{s2}\rangle$ basis states. What can you say about the state $|j = 2, m_j = 0\rangle$ using just the exchange operator?

   (c) Use orthogonality to write down the state $|j = 1, m_j = 1\rangle$ in terms of the $|m_{s1}, m_{s2}\rangle$ basis states. What is its P_{12} eigenvalue? Use this result to write down the state $|j = 1, m_j = 0\rangle$ in terms of the $|m_{s1}, m_{s2}\rangle$ basis states.

   (d) Based on your result for part (c), what can you say about the P_{12} eigenvalue of the state $|j = 0, m_j = 0\rangle$?

Formula sheet

Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t) = H\Psi(\vec{r},t) \tag{5}$$

Energy eigenstates: $H = \vec{p}^2/2m + V(\vec{r})$

$$H\psi(\vec{r}) = E\psi(\vec{r}), \qquad \Psi(\vec{r},t) = \psi(\vec{r})\,e^{-iEt/\hbar} \tag{6}$$

Differential operators corresponding to momentum and energy:

$$\vec{p} \to -i\hbar\vec{\nabla} \qquad E \to i\hbar\frac{\partial}{\partial t} \tag{7}$$

Harmonic oscillator in 1 dimension:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2, \qquad H|n\rangle = E_n|n\rangle, \qquad E_n = \hbar\omega(n+1/2) \tag{8}$$

Raising and lowering operators:

$$a|n\rangle = \sqrt{n}|n-1\rangle, \qquad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \tag{9}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger), \qquad p_x = -i\sqrt{\frac{m\omega\hbar}{2}}(a-a^\dagger) \tag{10}$$

Harmonic oscillator in 3 dimensions (isotropic), $H = \vec{p}^2/2m + m\omega^2\vec{r}^2/2 = H_x + H_y + H_z$: spatial wavefunctions are just the products of the solutions of the 1-dim harmonic oscillator.

$$H|n_1n_2n_3\rangle = E_{n_1n_2n_3}|n_1n_2n_3\rangle, \qquad E_{n_1n_2n_3} = \hbar\omega(n_1+n_2+n_3+3/2); \tag{11}$$

the wavefunctions can also be written in terms of the spherical harmonics,

$$\langle\vec{r}|n_r\ell m\rangle = R_{n_r\ell}(r)Y_{\ell m}(\theta,\phi), \qquad E_{n_r\ell m} = \hbar\omega(2n_r+\ell+3/2). \tag{12}$$

Particle in a box (infinite square well), 1 dimensional:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, \qquad 0 \le x \le L, \qquad E_n = \frac{\hbar^2\pi^2}{2mL^2}n^2 \tag{13}$$

Particle in a box, 3 dimensional, $L \times L \times L$, one corner at the origin:

$$\psi(x,y,z) = \left(\frac{2}{L}\right)^{3/2}\sin\frac{n_1\pi x}{L}\sin\frac{n_2\pi y}{L}\sin\frac{n_3\pi z}{L}, \qquad E = \frac{\hbar^2\pi^2}{2mL^2}(n_1^2+n_2^2+n_3^2) \tag{14}$$

Angular momentum:

$$L^2|\ell,m\rangle = \hbar^2\ell(\ell+1)|\ell,m\rangle, \qquad L_z|\ell,m\rangle = \hbar m|\ell,m\rangle, \qquad m_{\max} = \ell \tag{15}$$

$$L_\pm = L_x \pm iL_y, \qquad [L^2, L_\pm] = 0, \qquad L_\pm|\ell,m\rangle = C_\pm(\ell,m)|\ell,m\pm1\rangle \tag{16}$$

$$C_+(\ell,m) = \hbar\sqrt{(\ell-m)(\ell+m+1)}, \qquad C_-(\ell,m) = \hbar\sqrt{(\ell+m)(\ell-m+1)} \tag{17}$$

$$\vec{L} = \vec{r}\times\vec{p} \qquad \text{so, e.g., } L_z = xp_y - yp_x. \tag{18}$$

Addition of angular momentum: when $\vec{J} = \vec{L} + \vec{S}$,

$$j = \ell+s,\ldots,|\ell-s|, \qquad J_z = L_z + S_z, \qquad J_\pm = L_\pm + S_\pm \tag{19}$$

Spherical harmonics, up to $\ell = 2$:

$$Y_{00} = \frac{1}{\sqrt{4\pi}} \qquad Y_{\ell m}(-\hat{r}) = (-1)^\ell Y_{\ell m}(\hat{r})$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1,-1} = \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$$

$$Y_{22} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi} \qquad Y_{21} = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{i\phi} \qquad Y_{20} = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$$

$$Y_{2,-1} = \sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta e^{-i\phi} \qquad Y_{2,-2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-2i\phi} \tag{20}$$

Pauli principle: under the exchange of any two identical particles, multi-particle wavefunctions are symmetric for bosons (integer spin), antisymmetric for fermions (half-odd-integer spin).

Time-independent perturbation theory:

$$H = H_0 + \lambda H_1, \qquad H_0|\phi_n\rangle = E_n^{(0)}|\phi_n\rangle, \qquad H|\psi_n\rangle = E_n|\psi_n\rangle \tag{21}$$

$$E_n = E_n^{(0)} + E_n^{(1)} + E_n^{(2)} + \cdots \tag{22}$$

$$E_n^{(1)} = \langle\phi_n|\lambda H_1|\phi_n\rangle \qquad E_n^{(2)} = \sum_{k\neq n}\frac{|\langle\phi_n|\lambda H_1|\phi_k\rangle|^2}{E_n^{(0)} - E_k^{(0)}} \tag{23}$$

$$|\psi_n\rangle = |\phi_n\rangle + \sum_{k\neq n}\frac{\langle\phi_k|\lambda H_1|\phi_n\rangle}{E_n^{(0)} - E_k^{(0)}}|\phi_k\rangle + \mathcal{O}(\lambda^2) \tag{24}$$

Degenerate perturbation theory: first find the combinations of states that diagonalize the matrix $\langle\phi_i|\lambda H_1|\phi_j\rangle$ made up of states degenerate at zeroth order. The eigenvalues of this matrix are $E_n^{(1)}$.

Time-dependent perturbation theory:

$$H_0|\phi_n\rangle = E_n^{(0)}|\phi_n\rangle, \qquad i\hbar\frac{d}{dt}|\psi(t)\rangle = (H_0 + \lambda V(t))|\psi(t)\rangle \tag{25}$$

$$|\psi(t)\rangle = \sum_n c_n(t)e^{-iE_n^{(0)}t/\hbar}|\phi_n\rangle \tag{26}$$

For $c_k(0) = 1$ and all other $c_n(0) = 0$, to first order in λ the coefficients are

$$c_m(t) = \frac{1}{i\hbar}\int_0^t dt' e^{i\omega_{mk}t'}\langle\phi_m|\lambda V(t')|\phi_k\rangle \tag{27}$$

where $\omega_{mk} = (E_m^{(0)} - E_k^{(0)})/\hbar$. The transition probability is $P_{k\to m}(t) = |c_m(t)|^2$.

Variational principle: for *any* wavefunction $|\Psi\rangle$, $\langle\Psi|H|\Psi\rangle \geq E_0$ = ground state energy of H.

Some math:

Taylor series about $x = 0$:

$$f(x) = \sum_{n=0}^{\infty}\frac{1}{n!}\left.\frac{d^n f}{dx^n}\right|_{x=0} x^n \tag{28}$$

Eigenvalues of a matrix M (I is the unit matrix; solve for λ):

$$\det(M - \lambda \cdot I) = 0 \tag{29}$$

40. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 40.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).