Carleton University Physics Department

PHYS 4708 (Winter 2015, H. Logan)

Homework assignment #5

Handed out Wed. Feb. 25; due Wed. Mar. 4, 2015 at the start of class.
Problems are worth 5 points each unless noted otherwise.

1. Consider two noninteracting electrons in a one-dimensional harmonic oscillator with characteristic frequency ω. The spins can be arranged in a spin-singlet state ($j = 0$) or a spin-triplet state ($j = 1$).

   (a) Consider the spin-singlet state. What exchange-symmetry property must the spatial part of the two-particle ground state possess to satisfy the Pauli principle? Write down the properly normalized spatial part of the spin-singlet two-particle ground state (in terms of the single-particle states $|n\rangle$) and find its total energy.

   (b) Now consider the spin-triplet configuration. What exchange-symmetry property must the spatial part of the two-particle ground state possess to satisfy the Pauli principle? Write down the properly normalized spatial part of the spin-triplet two-particle ground state and find its total energy.

   (c) Work out the energies and degeneracies of the first five energy levels of this two-particle system, for both the spin-singlet and spin-triplet configurations.

2. Consider two identical non-interacting spin-3/2 particles.

   (a) Assume that the two particles are produced in an S-wave configuration, i.e., that their orbital angular momentum is zero. What are the values of the total spin $\vec{S}_{\rm tot} = \vec{S}_1 + \vec{S}_2$ that are allowed by the Pauli principle? (You can use the provided table of Clebsch-Gordan coefficients.)

   (b) Now assume that the two particles are produced in a P-wave configuration, i.e., that their orbital angular momentum is $\ell = 1$. What are the values of the total spin $\vec{S}_{\rm tot} = \vec{S}_1 + \vec{S}_2$ that are allowed by the Pauli principle? What are the allowed values of the total angular momentum (combining spin and orbital)?

continued....

3. *[10 points]* (Gasiorowicz 3rd ed. problem 15-3) Consider a particle in an infinite well in 1 dimension, with $V(x) = 0$ for $0 \leq x \leq a$ and $V(x) = \infty$ everywhere else. A tilt of the potential in the range $0 \leq x \leq a$ is turned on and then off according to

$$V_1(x,t) = \lambda \left(x - \frac{a}{2} \right) e^{-t^2/\tau^2}. \tag{1}$$

(a) Calculate the probability that a particle initially in the ground state ($n = 1$) ends up in the first excited state ($n = 2$).

(b) What is the probability that a particle initially in the ground state ends up in the second excited state ($n = 3$)?

(c) What happens to these results as $\tau \to \infty$? *(This shows that for a very slowly varying perturbation, transitions become strongly suppressed. The next problem gives a more general examination of this.)*

4. (Very similar to Gasiorowicz 3rd edition problem 15-6) This problem illustrates the *adiabatic theorem*. The theorem states that if the Hamiltonian is changed very slowly from H_0 to H, then a system in a given eigenstate of H_0 goes over into the corresponding eigenstate of H, but does not make any transitions. To be specific, consider the ground state, so that

$$H_0 \phi_0 = E_0 \phi_0. \tag{2}$$

Let $V(t) = V f(t)$, where $f(t)$ is a slowly varying function that interpolates monotonically from $f = 0$ at $t = 0$ to $f = 1$ as $t \to \infty$ as shown in the graph at the bottom of page 244 of the textbook. If the ground state of $H \equiv H_0 + V$ is $|w_0\rangle$, the theorem states that

$$|\langle w_0 | \psi(t) \rangle| \to 1 \tag{3}$$

as the time variation of f becomes infinitely slow.

(a) Show that

$$\frac{1}{i\hbar} \int_0^t dt' e^{i(E_m^0 - E_0^0)t'/\hbar} f(t') \to \frac{-e^{i(E_m^0 - E_0^0)t/\hbar}}{E_m^0 - E_0^0} \tag{4}$$

for times t such that $f(t) = 1$. Use the fact that

$$\frac{df(t')}{dt'} \ll \frac{E_m^0 - E_0^0}{\hbar} f(t'). \tag{5}$$

Note: it's most straightforward to use integration by parts; that is, to write

$$e^{i\omega t'} = \frac{1}{i\omega} \frac{d}{dt'} e^{i\omega t'}. \tag{6}$$

(b) Calculate $\psi(t)$ using time-dependent perturbation theory, equations (15-3) and (15-10) in the text. Compare this with the formula for the ground state of H as given by time-independent perturbation theory in equation (11-15), which here reads

$$|w_0\rangle = |\phi_0\rangle + \sum_{m \neq 0} \frac{\langle \phi_m | V | \phi_0 \rangle}{E_0^{(0)} - E_m^{(0)}} |\phi_m\rangle. \tag{7}$$

Thus show that, in the adiabatic limit,

$$|\psi(t)\rangle \to |w_0\rangle e^{-iE_0^{(0)} t/\hbar}. \tag{8}$$