The Symmetric Group

We will spend some time on the symmetric (or permutation) group and its representations.

- close link to important continuous groups
- relevant to systems of identical particles
- classification of tensors

Investigate useful results for the representations and develop a diagrammatic scheme called Young tableaux. This scheme can be applied to the unitary groups, which are often of physical significance.

The group $\mathcal{S}_n$ is the group of permutations of n objects. The order of $\mathcal{S}_n$ is $n!$. A useful concept is that of a cycle, a part of an arbitrary permutation which is independent of the rest. For instance the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 4 & 3 & 1 & 6 \end{pmatrix}$ can be written as the set of cycles $(125)(34)(6)$ or just $(125)(34)$. Here 1 is replaced by 2, 2 by 5, and 5 by 1, etc.

This example has 3 cycles of lengths 3, 2, and 1, which add up to $n = 6$. Thus a permutation of this type can be denoted as a partition $[l_1 l_2 l_3] = [321]$. In general a permutation can be denoted $[l_1 \ldots l_n]$ with l_i the cycle lengths in decreasing order ($l_{i+1} \leq l_i$) and $_i \sum l_i = n$.

The cycle structure is useful because all permutations with the same cycle structure belong to the same class. Thus, the number of classes of $\mathcal{S}_n$ and, hence, the number of irreducible representations is simply the number of different cycle structures possible (the number of ways n can be written as a sum of positive integers). See this as follows.

If two permutations P and P' are in the same class then there must exist a permutation Q that transforms P into P' as $P' = QPQ^{-1}$. Say

$$P = \begin{pmatrix} 1 & 2 & \ldots n \\ p_1 & p_2 & \ldots p_n \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1 & 2 & \ldots n \\ q_1 & q_2 & \ldots q_n \end{pmatrix} \equiv \begin{pmatrix} p_1 & p_2 & \ldots p_n \\ r_1 & r_2 & \ldots r_n \end{pmatrix}.$$

Then $QPQ^{-1} =$

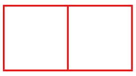
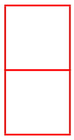
$$\begin{pmatrix} p_1 & p_2 & \ldots p_n \\ r_1 & r_2 & \ldots r_n \end{pmatrix}\begin{pmatrix} 1 & 2 & \ldots n \\ p_1 & p_2 & \ldots p_n \end{pmatrix}\begin{pmatrix} q_1 & q_2 & \ldots q_n \\ 1 & 2 & \ldots n \end{pmatrix}$$

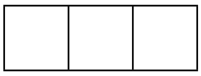
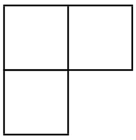
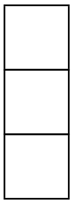
$$= \begin{pmatrix} q_1 & q_2 & \ldots q_n \\ r_1 & r_2 & \ldots r_n \end{pmatrix} = P'.$$

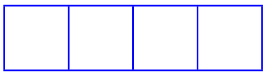
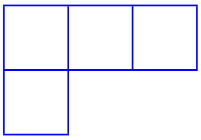
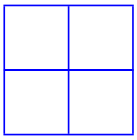
P' is just P with the top and bottom rows each permuted by Q. So if (125) is a closed cycle of P, then $(q_1 q_2 q_5)$ is a closed cycle of P' and the cycle structure remains intact.

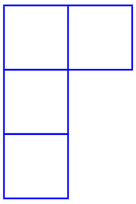
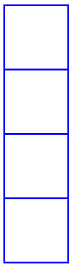
Thus, inequivalent irreps of $\mathcal{S}_n$ are determined by the distinct partitions of n into non-negative integers.

We may characterize these partitions by the so-called Young diagrams such that the pattern for the partition $n = l_1 + l_2 + \ldots + l_n$ has horizontal rows of $l_1, l_2, \ldots, l_n$ boxes, each row left justified. Some examples:

$\mathcal{S}_2$: $2 = 2 + 0$  $2 = 1 + 1$ 

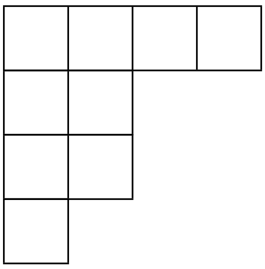
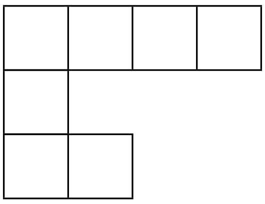
$\mathcal{S}_3$: $[3]$  $[21]$  $[111]$ 

$\mathcal{S}_4$: $[4]$  $[31]$  $[22]$ 

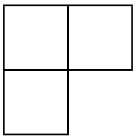
$[211]$  $[1111]$ 

Obviously no diagram will have more than n rows or columns. There are some other rules we will state.

Reading from top to bottom, the number of boxes in a row must not exceed the number of boxes in any previous row: $l_{i+1} \leq l_i$.

 OK  NOT OK

Reading from left to right, a column must not have its number of boxes larger than that of the column to its left.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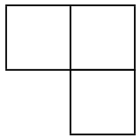
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These are just rules of notation.

Back to $\mathcal{S}_n$: It is useful to determine the one dimensional representations of the group. We will see that for any n, there are just two one-dimensional representations.

Any permutation (or cycle) can be written as a product of transpositions, permutations that interchange two objects. $P_{ij} = (ij)$ is a cycle of length 2. The breakdown of a cycle into transpositions is not unique. The number of transpositions is not even unique but will always be either even or odd for a given cycle. A cycle of odd length can be assigned, by definition, an even parity while one of even length will have an odd parity. The parity of a permutation is the product of parities of its cycles.

All transpositions must be in the same class since they all have the same cycle structure, the partition being $[211\ldots1]$. Thus all transpositions have the same character.

For a one dimensional representation, the character is the representation. All transpositions will be represented by the same number. Since the product of two identical transpositions is the identity $P_{ij}^2 = E$, the number representing transpositions must be ±1. So two possible 1-d repns exist: one for which all transpositions are represented by +1 and the other with transpositions represented by -1. In the former case, since any permutation is a product of transpositions, every permutation will be assigned +1, so this is the symmetric or identity representation. A function transforming according to this rep is unchanged by any permutation, so is totally symmetric. For the latter case, a permutation will be assigned +1 or -1 depending on whether it has even or odd parity. This is the anti-symmetric or alternating representation.

The parity of a permutation can be found using the following function

$$\phi_A = \begin{vmatrix} \phi_1(x_1) & \phi_2(x_1) \ldots & \phi_n(x_1) \\ \phi_1(x_2) & & \\ \vdots & & \\ \phi_1(x_n) & \phi_2(x_n) \ldots & \phi_n(x_n) \end{vmatrix}$$

A permutation of the variables x_i will interchange rows in this determinant, yielding a factor of ± 1 for even or odd interchanges - the parity. $P\phi_A = \epsilon_P \phi_A$. This function is a natural basis for the one-dimensional anti-symmetric representation because it transforms according to the parity - that is, it transforms according to the anti-symmetric representation. The function is called totally anti-symmetric since it has the property of changing sign under the transposition of any two elements:

$$P_{ij}\phi_A = -\phi_A$$

Now we can deduce the character table for $\mathcal{S}_n$.

We will discuss first the construction of a vector space invariant under permutations. Consider a partition $[n_1 n_2 \ldots]$ of $\mathcal{S}_n$. Consider a function

$$\phi([n_1 n_2 \ldots]) = \phi_1(1)\phi_1(2)\ldots\phi_1(n_1)\phi_2(n_1+1)\ldots \phi_2(n_2+n_1)\phi_3(n_1+n_2+1)\ldots$$

Basically we have n particles which are put into single particle states ϕ_i according to the partition. Now operating with the $n!$ permutations of $\mathcal{S}_n$ on $\phi([n_1 n_2 \ldots])$ will generate $\frac{n!}{n_1! n_2! \ldots}$ independent functions. This set provides a vector space which is invariant under $\mathcal{S}_n$, for each partition. Thus it provides a representation of $\mathcal{S}_n$ (not generally reducible).

We can now calculate the character for any permutation in the representation generated by the partition $[n_1 n_2 \ldots]$.

The character for any group element Q in the representation $[n_1 n_2 \ldots]$ is just the sum of its diagonal matrix elements in the linear vector space.

$$\sum_P \langle P\phi([n_1 n_2 \ldots])|Q|P\phi([n_1 n_2 \ldots])\rangle$$

But QP is just another permutation so $QP|\phi([n_1 n_2 \ldots])\rangle$ is another basis vector in the linear vector space. Thus the only contribution to the character will be for the states such that $Q(P\phi) = (P\phi)$ which are unchanged by Q. The character will be equal to the number of states unchanged by Q.

Notice the identity E is in the class (11), (111), (1111), etc for $\mathcal{S}_2$, $\mathcal{S}_3$, $\mathcal{S}_4$, etc respectively. For $Q = E$, all states are unchanged for this perm so the character is $\psi(E) = \frac{n!}{(n_1!n_2!...)}$. This is just the dimension of the vector space generated by the partition $[n_1 n_2 \ldots]$

Note the use of ψ here for character; the notation χ will be reserved for the irreducible representations.

We can consider a particular transposition to find the character of the class of transpositions. For the transposition $Q = P_{12}$, its character will be the number of basis states for which particles 1 and 2 are in the same single particle state. This is just the number of ways of rearranging the other $n-2$ particles:

$$\frac{(n-2)!}{(n_1-2)!n_2!n_3!\ldots} + \frac{(n-2)!}{n_1!(n_2-2)!n_3!\ldots} + \ldots$$

The first term corresponds to particles 1 and 2 in state ϕ_1 while the second has particles 1 and 2 in state ϕ_2, etc. For a permutation that is a single cycle of length l, this generalizes to

$$\frac{(n-l)!}{(n_1-l)!n_2!n_3!\ldots} + \frac{(n-l)!}{n_1!(n_2-l)!n_3!\ldots} + \ldots$$

If the permutation is a product of cycles the situation is more unwieldy. However, it is straightforward to deduce the general formula and use it in simple examples.

Of course the character for a permutation Q will vanish if the elements n_i of the partition $[n_1 n_2 \ldots]$ are too small to accomodate the cycles $(l_1 l_2 \ldots)$ of the class containing Q. (See character tables of $\mathcal{S}_{2,3,4}$.)

Notice also that the simple partition $[n]$ has character $\frac{n!}{n!} = 1$ for every class. This is to be expected as the partition $[n]$ has all the n particles in the same single particle state so the linear vector space is one dimensional. The function $\phi(1) \ldots \phi(n)$ is unchanged by any permutation. Thus this corresponds to the identity (symmetric) representation and is irreducible. So $\psi^{[n]} = \chi^{[n]} =$ the character of the identity representation.

We can proceed from this point to express the next level partition $[(n-1)1]$ in terms of $[n]$ and a remaining irreducible part by the standard methods introduced earlier in our study of representations and characters. That is, we can write the character of a representation as a sum of characters of irreps. For example, in $\mathcal{S}_4$,

$$\psi^{[31]} = \sum_\alpha m_\alpha \chi_p^\alpha.$$

Here the subscript p is a class label and the superscript α is a partition (irrep) label. For instance, $\alpha = 1$ corresponds to $[4]$. So,

$$\psi^{[31]} = m_1 \chi^{[4]} + \sum_{\alpha \neq 1} m_\alpha \chi_p^\alpha$$

By the orthogonality properties of characters of irreps, we can solve for the coefficient m_1 as

$$m_1 = \frac{1}{g} \sum_p c_p \chi_p^{[4]*} \psi^{[31]}$$

For $\mathcal{S}_4$, $g = 24$. After determining the c_p (i.e. the size of the classes), we simply plug in the known values of $\chi_p^{[4]} = 1, \forall p$ and $\psi^{[31]}$ from our tables to find the value $m_1 = 1$. Thus

$$\psi^{[31]} = \chi^{[4]} + \sum_{\alpha \neq 1} m_\alpha \chi_p^\alpha$$

Now we can show that what is left (after subtracting $\chi^{[4]}$) is actually irreducible and so we'll call it $\chi^{[31]}$.

After subtracting $\chi^{[4]} = 1$ from the character $\psi_p^{[31]}$ for each class p we have:

$$\psi_p^{[31]}/remainder \qquad 3 \qquad 1 \qquad -1 \qquad 0 \qquad -1$$

But, for an irrep, we have

$$\sum_p c_p \left|\chi_p\right|^2 = g$$

In fact, with the above results

$$\sum_p c_p \left|\psi_p^{[31]}/remainder\right|^2 = 24 = g$$

Thus, this remainder is itself irreducible.

You simply continue in this manner to decompose the next layer $\psi^{[22]}$ into irreps by solving for m_1 and m_2. The remainder will again be irreducible. We can continue this way through all the partitions.

At this point, we can now write down all possible irreps of $\mathcal{S}_n$ as all distinct partitions $[n_1 n_2 \ldots]$ and can also construct the character table for the irreps.

We will next consider a chain of subgroups of $\mathcal{S}_n$. We will see how Young diagrams can be a useful tool.

If we consider the permutation of objects labelled 1 through $n-1$ among n objects, then this will be the group $\mathcal{S}_{n-1}$ which must be a subgroup of $\mathcal{S}_n$. There will be a chain of subgroups:

$$\mathcal{S}_n \rightarrow \mathcal{S}_{n-1} \rightarrow \mathcal{S}_{n-2} \rightarrow \ldots \rightarrow \mathcal{S}_2.$$

An irrep of $\mathcal{S}_n$ will also provide a representation of $\mathcal{S}_{n-1}$, though not always an irrep. This follows from our previous study of subgroups.

Thus we should be able to reduce an irrep of $\mathcal{S}_n$ into a sum of irreps of $\mathcal{S}_{n-1}$ as

$$T^{(\alpha_n)} = \sum_{\alpha_{n-1}} m_{(\alpha_{n-1})} T^{(\alpha_{n-1})}$$

In the above, α_n is a partition of n, $T^{(\alpha_n)}$ is an irrep of $\mathcal{S}_n$, α_{n-1} is a partition of $n-1$, and $T^{(\alpha_{n-1})}$ is an irrep of $\mathcal{S}_{n-1}$.

Summing over diagonal elements, we obtain the equivalent relation for characters and we may then use the character table to determine the coefficients $m_{(\alpha_{n-1})}$:

$$m_{(\alpha_{n-1})} = \frac{1}{g} \sum_p c_p \chi_p^{(\alpha_{n-1})*} \chi_p^{\alpha_n}$$

The sum is over classes.

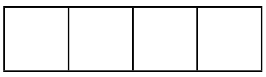
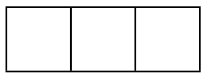
Only classes p of $\mathcal{S}_n$ with at least one cycle of length 1 can contribute. This is because at least one of the n objects must be unchanged by the permutation to form the subgroup $\mathcal{S}_{n-1}$.

Thus, for example, in the reduction of the partition $[4]$ of $\mathcal{S}_4$ to irreps of $\mathcal{S}_3$, we have:

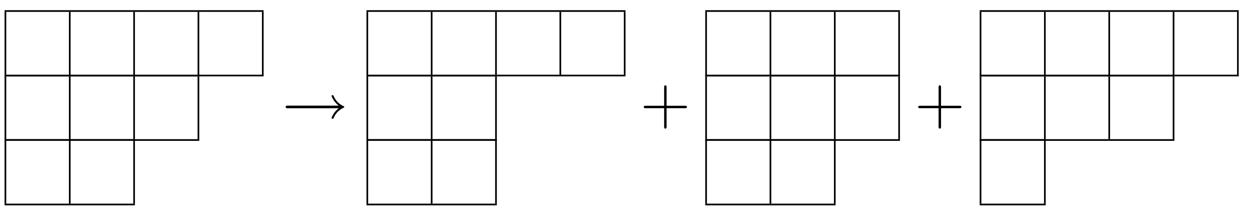
$$\chi^{[4]} = (1)\chi^{[3]} + (0)\chi^{[21]} + (0)\chi^{[111]}$$

Generally the coefficients $m_{(\alpha_{n-1})}$ are either zero or one. There is a simple way of determining which are nonzero with Young diagrams.

The irreps of $\mathcal{S}_{n-1}$ contained in the reduction of an irrep of $\mathcal{S}_n$ will be those corresponding to the Young diagrams obtained from the Young diagrams of $\mathcal{S}_n$ by removing one box.

So, indeed, as $\mathcal{S}_4 \longrightarrow \mathcal{S}_3$,  $\longrightarrow$ .

A more complicated example yields, as $\mathcal{S}_9 \longrightarrow \mathcal{S}_8$:



(At this point, we have not determined the above irreps.)

This reduction process can help in choosing basis vectors for an $\mathcal{S}_n$ invariant subspace. They can be chosen such that they transform according to an irrep of the subgroup $\mathcal{S}_{n-1}$. Similarly, irreps of $\mathcal{S}_{n-1}$ will decompose into irreps of $\mathcal{S}_{n-2}$, according to which the basis vectors can be chosen to transform. This chain continues down to $\mathcal{S}_2$.

Then the vectors spanning the irreps of $\mathcal{S}_n$ can be characterized by the chain of irreps of $\mathcal{S}_{n-1}, \mathcal{S}_{n-2}, \ldots, \mathcal{S}_2$ which they also span:

$$\alpha_n \quad \alpha_{n-1} \quad \alpha_{n-2} \ldots \alpha_2$$

where these are irreps of $\mathcal{S}_n \ldots \mathcal{S}_2$ respectively.

The chain of partitions can be represented by a single **Young tableau** with an appropriate labelling as follows. Enter the numbers 1 through n in the Young tableau for the partition α_n such that n is put into the square which, when removed, yields the $\mathcal{S}_{n-1}$ tableau α_{n-1}, and so on.

For instance the $\mathcal{S}_9$ example above could be broken down as:

$$\underset{\mathcal{S}_9}{\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&6&9\\\cline{1-3} 7&8\\\cline{1-2}\end{array}} \longrightarrow \underset{\mathcal{S}_8}{\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&6\\\cline{1-2} 7&8\\\cline{1-2}\end{array}} \longrightarrow \underset{\mathcal{S}_7}{\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&6\\\cline{1-2} 7\\\cline{1-1}\end{array}} \longrightarrow \underset{\mathcal{S}_6}{\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&6\\\cline{1-2}\end{array}} \longrightarrow$$

$$\longrightarrow \underset{\mathcal{S}_5}{\begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3\\\cline{1-1}\end{array}} \longrightarrow \underset{\mathcal{S}_4}{\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}} \longrightarrow \underset{\mathcal{S}_3}{\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}} \longrightarrow \underset{\mathcal{S}_2}{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}}$$

There are many ways to proceed through this chain. The number of ways in which the integers can be placed in the boxes in this 'natural order' (increasing left to right and top to bottom while allowing only legal Young Tableaux) is the dimension of the irrep of $\mathcal{S}_n$. (Recall the dimension is given in the character table as the character of the identity element.)

For the simple case of $\mathcal{S}_3$, the following labels are allowed:

$[3]$ is the 1-d symmetric irrep: $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array}$

$[21]$ is the 2-d irrep: $\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ $\quad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$

$[111]$ is the 1-d anti-symmetric irrep: $\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$

We will next develop a couple ways of multiplying representations. First consider the direct product of two representations. We have already studied this in the general context of representation theory.

Recall that a direct product of two irreps of $\mathcal{S}_n$ can be reduced to a sum over irreps of $\mathcal{S}_n$

$$T^\alpha \times T^\beta = \sum_\gamma m_\gamma T^\gamma$$

with coefficients

$$m_\gamma = \frac{1}{g} \sum_p c_p \chi_p^{\gamma *} \chi_p^\alpha \chi_p^\beta$$

given from the character tables.

In order to introduce the concept of conjugate or adjoint representation, consider the direct product of an irrep T^α with the one dimensional totally antisymmetric irrep

$[111\ldots1]$. The character, for an element in a given class, of $T^\alpha \times T^\beta$ is just the product of the characters, for that class, of T^α and T^β. But the characters for $[111\ldots1]$ are just the parities of each class, $\epsilon_p = \pm 1$. So the characters of our direct product representation will be $\chi_p^{\alpha\times[11\ldots1]} = \epsilon_p \chi_p^\alpha$ for elements in class p. So, summing over classes,

$$\sum_p c_p \left|\chi_p^{\alpha\times[11\ldots1]}\right|^2 = \sum_p c_p |\epsilon_p|^2 |\chi_p^\alpha|^2 = g$$

since $|\epsilon_p|^2 = 1$ and T^α is an irrep. Thus, since it saturates this sum, $T^\alpha \times T^{[11\ldots1]}$ is also irreducible. Denote this product representation as $T^{\tilde{\alpha}}$. We see from the character tables that $\tilde{\alpha}$ is related to α by the interchange of rows and columns in their Young diagrams. That is, if in $\mathcal{S}_4$, $\alpha = [31] = \ydiagram{3,1}$, then $\tilde{\alpha} = [211] = \ydiagram{2,1,1}$.

$\tilde{\alpha}$ is called the conjugate or adjoint representation to α.

Some irreps are self-adjoint; for these, the characters of any class with odd parity vanish. This is because T^α and $T^\alpha \times T^{[11\ldots1]}$ are equivalent in this case and so must have the same characters. The only solution to $(-1)\chi_p^\alpha = \chi_p^\alpha$ is $\chi_p^\alpha = 0$.

We can also check for the occurrence of the symmetric $[n]$ and the anti-symmetric $[111\ldots1]$ representations in direct product expansions. We have

$$m_{[n]} = \frac{1}{g}\sum_p c_p \chi_p^{[n]*}\chi_p^\alpha \chi_p^\beta = \delta_{\alpha\beta}$$

since $\chi_p^{[n]} = 1$. This means the symmetric representation can only occur if one takes the direct product of two equivalent irreps (i.e. the same characters). Also

$$m_{[11\ldots1]} = \frac{1}{g}\sum_p c_p \epsilon_p \chi_p^\alpha \chi_p^\beta = \frac{1}{g}\sum_p c_p \chi_p^{\tilde{\alpha}} \chi_p^\beta = \delta_{\tilde{\alpha}\beta}$$

since $\chi_p^{[11\ldots1]} = \epsilon_p$. The anti-symmetric irrep will only be produced in the direct product of an irrep with its adjoint. These rules are obviously of importance if one needs to form a state with specific symmetry properties.

Now define another type of product representation - the outer product. Here we use an irrep T^{α} of $\mathcal{S}_n$ and an irrep $T^{\alpha'}$ of $\mathcal{S}_{n'}$ to generate a representation T of $\mathcal{S}_{n+n'}$, which will then be reduced into irreps of $\mathcal{S}_{n+n'}$.

$$T = T^{\alpha} \otimes T^{\alpha'}$$

One can construct products of functions $f_i(1, 2, \ldots, n)$ and $g_j(n+1, \ldots, n+n')$ that transform according to irreps T^{α} of $\mathcal{S}_n$ and $T^{\alpha'}$ of $\mathcal{S}_{n'}$, respectively. Here $i = 1, 2, \ldots, s_{\alpha}$ and $j = 1, 2, \ldots, s'_{\alpha}$. Those $s_{\alpha}s_{\alpha'}$ products $f_i g_j$ will not form an invariant space under $\mathcal{S}_{n+n'}$ but, allowing for all permutations between the n particles in f and the n' in g will complete the space.

The dimension of the space will be

$$s_{\alpha}s_{\alpha'}\frac{(n+n')!}{n!n'!}$$

where

- s_α is the dimension of the space for T^α in $\mathcal{S}_n$
- $s_{\alpha'}$ is the dimension of the space for $T^{\alpha'}$ in $\mathcal{S}_{n'}$
- $\frac{(n+n')!}{n!n'!}$ is the number of ways of choosing n objects in $n + n'$

So T will provide a (generally reducible) representation of $\mathcal{S}_{n+n'}$.

We can see how this reduces to a sum of irreps by first considering outer products of a representation of $\mathcal{S}_n$ with the irrep of $\mathcal{S}_1$ $\square$, $n' = 1$.

First consider $n = n' = 1$. In this case $\square \otimes \square$ will yield a two-dimensional reducible representation of $\mathcal{S}_2$. The corresponding functions providing a basis would include $f(1)g(2)$ and the permuted function $f(2)g(1)$. These can be combined, however, into the (1-d) symmetric and antisymmetric irreps ($f(1)g(2) \pm f(2)g(1)$) of $\mathcal{S}_2$ as:

$$\square \otimes \square = \square\!\square + \begin{array}{c}\square\\[-4pt]\square\end{array}$$

Now building on this, consider $\ydiagram{2} \otimes \ydiagram{1}$. $\ydiagram{2}$ is symmetric in its particles so only $\ydiagram{3}$ and $\ydiagram{2,1}$ can be formed in this product. These are the only allowed ways a single box can be added to the Young diagram $\ydiagram{2}$. Thus the resulting 3-dimensional space will consist of the totally symmetric basis vector for $\ydiagram{3}$ and the two mixed-symmetric basis vectors of $\ydiagram{2,1}$. In terms of the functions $f(1,2)g(3)$ and their permutations, we can write the symmetric function $f(1,2) = \phi(1)\phi(2)$. The permutations are then $\phi(1)\phi(2)g(3)$, $\phi(1)g(2)\phi(3)$, and $g(1)\phi(2)\phi(3)$. The completely symmetric combination (ignoring normalization)

$$\phi(1)\phi(2)g(3) + \phi(1)g(2)\phi(3) + g(1)\phi(2)\phi(3)$$

provides the basis for the 1-d irrep $\ytableau{1 & 2 & 3}$. The remaining combinations constitute the Young tableaux $\ytableau{1 & 2 \\ 3}$

and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$ and can be written in terms of functions

$$2\phi(1)\phi(2)g(3) - \phi(1)g(2)\phi(3) - g(1)\phi(2)\phi(3)$$

and

$$\phi(1)g(2)\phi(3) - g(1)\phi(2)\phi(3).$$

Generally, the outer product of T^α with $\square$ yields

$$T^\alpha \otimes \square = \sum_\beta T^\beta$$

where the sum is over all tableaux formed by adding a box to the tableau for T^α. We will now simply state the rules for the general case.

$$T = T^\alpha \otimes T^{\alpha'} = \sum_\beta m(\beta, \alpha, \alpha')T^\beta \qquad (*)$$

In the above α is a partition of n, α' is a partition of n', and β is a partition of $n + n'$. $m(\beta, \alpha, \alpha')$ is the multiplicity of partition β in the product. These partitions are formed as follows.

1. Place a letter a in each box of the first row of tableau α', b in each box of the second row, etc.

$$\alpha' = \begin{array}{|c|c|c|} \hline a & a & a \\ \hline b & b & \\ \cline{1-2} c & & \\ \cline{1-1} \end{array}$$

   This shows the symmetry within rows.

2. Add the boxes of α' to the tableau for α such that the a's are added first, the b's next, etc. and such that the result is an allowed Young tableau, with no identical letters appearing in a column. (This is because identical letters are indicating a symmetric state.) There cannot be fewer a's in reading from right to left than b's, etc.

All the resulting diagrams will yield the sum above. These are known as Littlewood-Richardson rules. Note that the maximum number of rows possible is $n + n'$.

The following example gives the outer product of the partition $[22]$ of $\mathcal{S}_4$ with the partition $[21]$ of $\mathcal{S}_3$ to yield a sum of partitions of $\mathcal{S}_7$.

$$\ydiagram{2,2} \otimes \begin{ytableau} a & a \\ b \end{ytableau} = \begin{ytableau} \ & \ & a & a \\ \ & \ & b \end{ytableau} + \begin{ytableau} \ & \ & a & a \\ \ & \ \\ b \end{ytableau} + \begin{ytableau} \ & \ & a \\ \ & \ \\ a \\ b \end{ytableau} + \begin{ytableau} \ & \ & a \\ \ & \ \\ a & b \end{ytableau}$$

$$+ \begin{ytableau} \ & \ & a \\ \ & \ & b \\ a \end{ytableau} + \begin{ytableau} \ & \ \\ \ & \ \\ a & a \\ b \end{ytableau}$$

We can see that this works dimensionally.

$$\frac{2 \times 2 \times 7!}{4!3!} = 140 = 14 + 35 + 35 + 21 + 21 + 14$$

$$\begin{aligned} T^{[22]} \otimes T^{[21]} = & \\ & T^{[43]} + T^{[421]} + T^{[3211]} + \\ & T^{[322]} + T^{[331]} + T^{[2221]} \end{aligned}$$

We have noted that the dimension of the irreps is given by the number of allowed ways to put numbers into the Young tableaux. A general formula for the dimensions of the irreps was developed by Frobenius. Details are given in Hamermesh.

However, there is a simple formula that is typically used in particle physics, known as the hook rule. By the hook rule, the dimension of an irrep of $\mathcal{S}_n$ corresponding to a given partition is the ratio of the order of the group $n!$ to the product of the hook lengths. Each box in the Young diagram is assigned a hook length $h = (r + b + 1)$ where r is the number of boxes to the right of that box in the same row and b is the number of boxes below that box in the same column.

$$s_{[n_1 n_2 \ldots]} = \frac{n!}{\prod_i h_i}$$

The hook length is the number of boxes one passes through coming in from the right along a row to the box in question, hooking down $90°$, and going down along a row.

For the first irrep in the reduction above, the hook lengths are given in each box as $\begin{array}{|c|c|c|c|}\hline 5 & 4 & 3 & 1 \\ \hline 3 & 2 & 1 \\ \cline{1-3}\end{array}$ such that the dimension is $\frac{7!}{5\cdot 4\cdot 3\cdot 1\cdot 3\cdot 2\cdot 1} = 14$.

Now look at the reduction of irreps of $\mathcal{S}_{n+n'}$ under the subgroup $\mathcal{S}_n \times \mathcal{S}_{n'}$. Since $\mathcal{S}_n \times \mathcal{S}_{n'}$, with $\mathcal{S}_n$ referring to labels $1 \to n$ and $\mathcal{S}_{n'}$ referring to labels $n+1 \to n+n'$, is a subgroup of $\mathcal{S}_{n+n'}$, we can reduce an irrep T^β of $\mathcal{S}_{n+n'}$ as

$$T^\beta = \sum_{\alpha,\alpha'} m(\beta,\alpha,\alpha') T^{\alpha\times\alpha'} \qquad (**)$$

Recall the relation $\chi^{\alpha\times\beta}(G_a H_b) = \chi^\alpha(G_a)\chi^\beta(H_b)$ for characters of a product group, where G_a is an element of $\mathcal{S}_n$ and H_b of $\mathcal{S}_{n'}$.

Summing the representation relation $(**)$ above over diagonal elements, it becomes a relation for characters

$$\chi^\beta = \sum_{\alpha,\alpha'} m(\beta,\alpha,\alpha') \chi^{\alpha\times\alpha'}$$

Using the orthogonality of irreps of $\mathcal{S}_n \times \mathcal{S}_{n'}$ yields

$$m(\beta,\alpha,\alpha') = \frac{1}{n!n'!} \sum_{G,H} \chi^\beta(GH)\chi^{\alpha\times\alpha'}(GH)$$

Notice the coefficients of equation $(*)$ on page 25 for the reduction of the outer product into a sum of irreps of $\mathcal{S}_{n+n'}$ are given by

$$m(\beta,\alpha,\alpha') = \frac{1}{(n+n')!}\sum_B \chi^\beta(B)\chi(B)$$

where B is in $\mathcal{S}_{n+n'}$ and $\chi(B)$ is the character of $T^\alpha \otimes T^{\alpha'} = T$, the outer product representation. Since T was generated by functions

$$f_i(1,2,\ldots,n)g_j(n+1,\ldots,n+n'),$$

any class in $\mathcal{S}_{n+n'}$ whose elements are not in the subgroup $\mathcal{S}_n \times \mathcal{S}_{n'}$ must have $\chi = 0$ in T. For the remaining elements of $\mathcal{S}_{n+n'}$, the character is given by

$$c_B\chi(B) = \frac{(n+n')!}{n!n'!}c_{GH}\chi^{\alpha\times\alpha'}(GH)$$

GH is an element of $\mathcal{S}_n \times \mathcal{S}_{n'}$ belonging to the same class as B. c_B is the number of elements in the corresponding class of $\mathcal{S}_{n+n'}$ and c_{GH} is the number of elements in the corresponding class of $\mathcal{S}_n \times \mathcal{S}_{n'}$. Thus the coefficients in equation $(*)$ on page 25 and $(**)$ on page 29 are the same. So one can construct the character tables of product groups from existing tables and then reduce the group $\mathcal{S}_{n+n'}$ into a sum of irreps of $\mathcal{S}_n \times \mathcal{S}_{n'}$.

The reduction of irreps of $\mathcal{S}_4$ into irreps of the subgroup $\mathcal{S}_2 \times \mathcal{S}_2$ follows as an example. First form the character table of $\mathcal{S}_2 \times \mathcal{S}_2$. (see the handout) Use that along with the character table of $\mathcal{S}_4$ to find:

$$\begin{aligned}
\cdot[4] &= [2] \times [2] \\
\cdot[31] &= [2] \times [2] + [2] \times [11] + [11] \times [2] \\
\cdot[22] &= [2] \times [2] + [11] \times [11] \\
\cdot[211] &= [2] \times [11] + [11] \times [2] + [11] \times [11] \\
\cdot[1111] &= [11] \times [11]
\end{aligned}$$

The class size c_p for the symmetric group (Hamermesh)

We know that the size of any class of $\mathcal{S}_n$ is the number of distinct permutations that share the same cycle structure.

For a class p, there is a systematic way to calculate this size, c_p, by characterizing the cycle structure according to the number of 1-cycles (p_1), the number of 2-cycles (p_2), and so on.

To start, the total of n objects can be permuted $n!$ ways.

However, these are not all distinct. First, the cycles of a particular length can be put in any order - for instance, for the 1-cycles, (1)(2) is the same as (2)(1), so there are $p_1!$ ways to arrange these 1-cycles. The same holds for any of the other cycles - (12)(34) is the same as (34)(12). So among the $n!$ permutations, $p_1!p_2!\ldots p_n!$ are redundant.

Similarly, within each occurence of a particular cycle, cycling through its entries does not yield a distinct permutation. As examples, (i) the 2-cycle (12) is the same as

(21), so one must divide by a factor of 2 for each 2-cycle (that is, by 2^{p_2} and (ii) the 3-cycles (123), (231), and (312) are all the same so divide by 3^{p_3}, and so on. This redundancy factor is $2^{p_2} \dots n^{p_n}$.

This exhausts all the redundancies yielding the class size as

$$c_p = \frac{n!}{p_1! \cdot 2^{p_2} \cdot p_2! \cdot 3^{p_3} \cdot p_3! \dots n^{p_n} \cdot p_n!}.$$